ANALYSIS OF A METHOD TO PARAMETERIZE PLANAR CURVES IMMERSED IN TRIANGULATIONS

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Abstract. We prove that a planar C^2-regular boundary Γ can always be parameterized with its closest point projection π over a certain collection of edges Γ_h in an ambient triangulation, by making simple assumptions on the background mesh. For Γ_h, we select the edges that have both vertices on one side of Γ and belong to a triangle that has a vertex on the other side. By assuming a quasi-uniform family of background meshes, a sufficiently small mesh size h and that certain angles in each mesh are acute, we prove that $\pi : \Gamma_h \to \Gamma$ is a homeomorphism and that it is C^1 on each edge in Γ_h. We provide bounds for the Jacobian of the parameterization and local estimates for the required mesh size, which could be used in adapting the ambient triangulation. Such a parameterization was first proposed in [17] where it was applied to the construction of a high-order immersed boundary method on a class of planar piecewise C^2-curves.

Key words. curve parameterization; closest point projection; immersed boundary

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1. Introduction. The purpose of this article is to analyze a method to parameterize planar C^2-regular boundaries over a collection of edges in a background triangulation. Such a parameterization was introduced by the authors in [17]. The method consists in making specific choices for the edges in the background mesh and for the map from these edges onto the curve. For the edges, we select the ones that have both vertices on one side of the (orientable) curve to be parameterized and belong to a triangle that has a vertex on the other side. Such edges are termed positive edges. For the map, we select the closest point projection of the curve. In this article, we prove that the closest point projection restricted to the collection of positive edges is a homeomorphism onto the curve and that it is C^1 on each positive edge (Theorem 3.1). For this, we have to impose restrictions on the family of background meshes; we require a sufficiently small mesh size, quasi-uniformity, and that certain angles in each mesh be acute and away from 90° by a value independent of the mesh size. We also provide a computable *a priori* estimate for the required mesh size as well as bounds for the Jacobian of the computed parameterization for the curve.

One of the main motivations behind the parameterization analyzed here is to accurately represent planar curved domains over non-conforming background meshes. For once the curved boundary is parameterized over a collection of nearby edges, a suitable collection of triangles in the background mesh can be mapped to curved ones to yield an accurate spatial discretization for the curved domain. Constructing mappings from straight triangles to curved ones and their analysis has been the subject of numerous articles by a notable list of authors; we refer to a representative few [5, 7, 9, 14, 16, 19, 20, 22] for details on this subject. Almost without exception, these constructions have two assumptions in common: (i) a mesh with edges that interpolate the curved boundary and (ii) a (local) parametric representation for the curved boundary. The former entails careful mesh generation while the latter limits how the boundary can be described.

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The parameterization analyzed here relaxes both these assumptions. Indeed, a compelling consequence of Theorem 3.1 is that any planar smooth boundary can be parameterized with its closest point projection over the collection of positive edges in any sufficiently refined background mesh of equilateral triangles. It is also interesting to note that the theorem does not guarantee the same with a background mesh of right angled triangles. The latter family of meshes may not satisfy the required assumption on angles, see (3.1) in Theorem 3.1. On a related note, in [17] we described a way of parameterizing curves over edges and diagonals of meshes containing just parallelograms, in particular structured meshes of rectangles. See also [6] for a triangulation algorithm with a similar objective.

That edges in the same background mesh can be used to parameterize a large family of planar curves is a valuable result in the context of Immersed Boundary Methods (IBM). These are numerical schemes that do not require a conforming spatial discretization for the problem domain. Most IBMs in the literature approximate the boundary with piecewise straight line segments. Since the resulting errors are of the right order of the mesh size only for low order approximations of the solution (piecewise constant/linear, cf., [15, 18]), high order IBMs (that adopt high order interpolations for the solution) are quite rare in the literature. In [17], we described a high order IBM for problems on curves by adopting the parameterization analyzed here and demonstrated its optimal convergence properties with numerical examples. In future articles, we intend to extend such a construction to create optimally convergent high order IBMs for problems over curved domains. The parameterization is also ideally suited for the finite element method with p-refinement (see [4, 8, 21]). Once the background mesh is sufficiently refined, the same mesh can be used for calculations with progressively higher order interpolations.

We anticipate the parameterization to be a useful tool in the numerical solution of an assortment of problems. First, in problems that are sensitive to perturbations in the boundary and boundary conditions. Problems of this nature include for example, the "Babuška paradox" related to plate bending problems [3]. Another class consists of problems with evolving boundaries/interfaces. For at least in principle, the same background mesh could be used to describe the domain instead of repeatedly remeshing as the domain evolves. Additionally, when the same mesh is used, the sparsity structure of matrices such as the mass and stiffness can be retained. We are particularly interested in hydraulic fracture problems. These involve solving a coupled system of partial differential equations, the one for the (poro)elastic response of a solid, and the lubrication equation posed over evolving fluid-filled cracks [1].

The parameterization is very easy to implement (in parallel) and independent of the particular description adopted for the curve. It also extends naturally to planar curves with corners, self-intersections, T-junctions and practically all planar curves of interest in engineering and computer graphics applications. The idea is to construct such curves by splicing end-to-end, arcs of C^2-regular boundaries and parameterize each arc with the method described here. We refer the reader to [17] for details on this and for discussions on implementation.

An outline of the proof of Theorem 3.1 is given in §3.3. The crux of the proof is demonstrating injectivity of the parameterization over the collection of positive edges (Γ_h). Regularity of the parameterization and estimates for the Jacobian follow easily from regularity of the curve (Γ) and some straightforward calculations. We prove injectivity by inspecting the restriction to each positive edge, then to pairs of intersecting positive edges, and finally to connected components of Γ_h, of the closest

point projection (π) to Γ. The assumption that certain angles in each mesh are acute has a very simple geometric motivation (see Fig. 3.1) and ensures injectivity over each positive edge (§4,5). Extending this to the entire set Γ_h is non-trivial, requiring some careful, albeit simple topological arguments. It entails understanding how and how many positive edges intersect at each vertex in Γ_h leading us to show in §6 that each connected component of Γ_h is a Jordan curve. We then show in §7 that the restriction of π to each connected component of Γ_h is a parameterization of a connected component of Γ. Finally in §8, we establish a correspondence between connected components of Γ and Γ_h.

2. Preliminary definitions. In order to state our main result with the requisite assumptions, a few definitions are essential. First, we define the family of planar C^2-regular boundaries, the curves we consider for parameterization.

Definition 2.1 ([11, def. 1.2]). *A bounded open set $\Omega \subset \mathbb{R}^2$ has a C^2-regular boundary if there exists $\Psi \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $\Omega = \{x \in \mathbb{R}^2 : \Psi(x) < 0\}$ and $\Psi(x) = 0$ implies $|\nabla\Psi| \geq 1$. We say that Ω is C^2-regular domain and that $\partial\Omega$ is a C^2-regular boundary. The function Ψ is called a defining function for Ω.*

There are in fact a few equivalent notions of C^2-regular boundaries (and more generally C^k-regular boundaries), see [12]. For future reference, we note that each connected component of a C^2-regular boundary is a Jordan curve with bounded curvature.

Recall the definitions of the signed distance function and the closest point projection for a curve Γ that is the boundary of an open and bounded set Ω in $\mathbb{R}^2$. The signed distance to Γ is the map $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as $-\min_{y\in\Gamma} d(\cdot, y)$ over Ω and as $\min_{y\in\Gamma} d(\cdot, y)$ elsewhere. The function $d(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R}^2$. The closest point projection π onto Γ is the map $\pi : \mathbb{R}^2 \to \Gamma$ given by $\pi(\cdot) = \arg\min_{y\in\Gamma} d(\cdot, y)$.

The following theorem quoted from [11] is a vital result for our analysis. It concerns the regularity of the signed distance function ϕ and closest point projection π for a C^2-regular boundary. The theorem also shows that ϕ is a defining function for a C^2-regular domain. In the statement, the ε-ball centered at $x \in \mathbb{R}^2$ is $B(x, \varepsilon) := \{y : d(x, y) < \varepsilon\}$ and the ε-neighborhood of $A \subset \mathbb{R}^2$ is the set $B(A, \varepsilon) := \cup_{x\in A} B(x, \varepsilon)$.

Theorem 2.2 ([11, Theorem 1.5]). *If $\Omega \subset \mathbb{R}^2$ is an open set with a C^2-regular boundary, then there exists $r_n > 0$ such that $\phi : B(\partial\Omega, r_n) \to (-r_n, r_n)$ and $\pi : B(\partial\Omega, r_n) \to \partial\Omega$ are well defined. The map ϕ is C^2 while π is a C^1 retraction onto $\partial\Omega$. The mapping $x \mapsto (\phi(x), \pi(x)) : B(\partial\Omega, r_n) \to (-r_n, r_n) \times \partial\Omega$ is a C^1-diffeomorphism with inverse $(\phi, \xi) \mapsto \xi + \phi\hat{N}(\xi) : (-r_n, r_n) \times \partial\Omega \to B(\partial\Omega, r_n)$ where $\hat{N}(\xi)$ is the unit outward normal to $\partial\Omega$ at ξ. Furthermore, ϕ is the unique solution of $|\nabla\phi| = 1$ in $B(\partial\Omega, r_n)$ with $\phi = 0$ on $\partial\Omega$ and $\nabla\phi \cdot \hat{N} > 0$ on $\partial\Omega$.*

In Theorem 2.2, by saying that ϕ and π are well defined over $B(\partial\Omega, r_n)$, we mean that these maps are defined and have a unique value at each point in $B(\partial\Omega, r_n)$. The following proposition follows from [10, §14.6]. A simple derivation specific to planar curves can be found in [17].

Proposition 2.3. *Let $\Gamma \subset \mathbb{R}^2$ be a C^2-regular boundary with signed distance function ϕ, closest point projection π and signed curvature κ_s. If $p \in B(\Gamma, r_n)$ and $|\phi(p)\kappa_s(\pi(p))| < 1$, then*

$$\nabla\pi(p) = \frac{\hat{T}(\pi(p)) \otimes \hat{T}(\pi(p))}{1 - \phi(p)\,\kappa_s(\pi(p))}, \tag{2.1a}$$

$$\textit{and} \quad \nabla\nabla\phi(p) = -\kappa_s(\pi(p))\nabla\pi(p). \tag{2.1b}$$

For parameterizing C^2-regular boundaries, we will consider a family of background meshes that are triangulations of polygonal domains (cf., [13, chapter 4]). We mention the related terminology and notation used in the remainder of the article. With each triangulation $\mathcal{T}_h$, we associate a pairing (V, C) of a vertex list V that is a finite set of points in $\mathbb{R}^2$ and a connectivity table C that is a collection of ordered 3-tuples in $V \times V \times V$ modulo permutations. A vertex in $\mathcal{T}_h$ is thus an element of V (and hence a point in $\mathbb{R}^2$). An edge in $\mathcal{T}_h$ is a closed line segment joining two vertices of a member of C. The relative interior of an edge e_{pq} with end points (or vertices) p and q is the set $\mathbf{ri}\,(e_{pq}) = e_{pq} \setminus \{p, q\}$.

A triangle K in $\mathcal{T}_h$, denoted $K \in \mathcal{T}_h$, is the interior of the triangle in $\mathbb{R}^2$ with vertices given by its connectivity $\hat{K} \in C$. Frequently, we will not distinguish between the triangle K and its connectivity $\hat{K}$ unless the distinction is essential. We refer to the diameter of K by h_K and the diameter of the largest ball contained in $\overline{K}$ by ρ_K. The ratio $\sigma_K := h_K/\rho_K$ is called the shape parameter of K (see [13, chapter 3]). Later, we will invoke the fact that $\sigma_K \geq \sqrt{3}$ with equality holding for equilateral triangles. A family of triangulations $\{\mathcal{T}_h\}_h$ is quasi-uniform if for each h, $\mathcal{T}_h$ is a triangulation of a polygon, and there exist constants $\sigma > 0$ and $\tau > 0$ such that for each h and $K \in \mathcal{T}_h$,

$$\sigma_K = \frac{h_K}{\rho_K} \leq \sigma \quad \text{and} \quad \frac{h_K}{h} \geq \tau. \tag{2.2}$$

To consider curves immersed in background triangulations, we introduce the following terminology.

Definition 2.4. *Let $\Gamma \subset \mathbb{R}^2$ be a C^2-regular boundary with signed distance function ϕ and let $\mathcal{T}_h$ be a triangulation of a polygon in $\mathbb{R}^2$.*

(i) *We say that Γ is immersed in $\mathcal{T}_h$ if $\Gamma \subset \mathrm{int}\left(\cup_{K\in\mathcal{T}_h} \overline{K}\right)$.*

(ii) *A triangle in $\mathcal{T}_h$ is positively cut by Γ if $\phi \geq 0$ at precisely two of its vertices.*

(iii) *An edge in $\mathcal{T}_h$ is a positive edge if $\phi \geq 0$ at both of its vertices and if it is an edge of a triangle that is positively cut by Γ.*

(iv) *The proximal vertex of a triangle positively cut by Γ is the vertex of its positive edge closest to Γ. When both vertices of the positive edge are equidistant from Γ, either one can be designated to be the proximal vertex.*

(v) *The conditioning angle of a triangle positively cut by Γ is the interior angle at its proximal vertex.*

3. Main result.

The main result of this article is the following.

Theorem 3.1. *Consider a C^2-regular boundary $\Gamma \subset \mathbb{R}^2$ with signed distance function ϕ, closest point projection π and curvature κ. Let $\{\mathcal{T}_h\}_h$ be a quasi-uniform family of triangulations such that Γ is immersed in $\mathcal{T}_h$ for each h. Denote the union of positive edges in $\mathcal{T}_h$ by Γ_h, the collection of triangles positively cut by Γ in $\mathcal{T}_h$ by $\mathcal{P}_h$ and the conditioning angle of $K \in \mathcal{P}_h$ by ϑ_K. If*

$$\Theta_c := \sup_h \max_{K\in\mathcal{P}_h} \vartheta_K < 90^\circ, \tag{3.1}$$

then there exists $h_0 > 0$ such that for any $h < h_0$,

(i) *each positive edge in Γ_h is an edge of precisely one triangle in $\mathcal{P}_h$,*

(ii) *for each positive edge $e \subset \Gamma_h$, π is a C^1-diffeomorphism over $\boldsymbol{ri}(e)$,*

(iii) *if $K = (p, q, r) \in \mathcal{P}_h$ has positive edge e_{pq}, then*

$$-\,C_K^h h_K^2 < \phi(x) \leq h_K \quad \forall x \in e_{pq}, \tag{3.2}$$

$$\textit{where}\quad C_K^h := \frac{M_K}{1 - M_K h_K} \quad \textit{and} \quad M_K := \max_{B(K,h_K)\cap\Gamma} \kappa.$$

The Jacobian J of the map $\pi : \boldsymbol{ri}(e_{pq}) \to \Gamma$ satisfies

$$\frac{\sin(\beta_K - \vartheta_K)}{1 + M_K h_K} \le J(x) = \left|\nabla\pi(x) \cdot \frac{(p-q)}{d(p,q)}\right| \le \frac{1}{1 - M_K h_K} \quad \forall x \in \boldsymbol{ri}(e_{pq}), \tag{3.3}$$

where

$$\cos\beta_K := C_K^h \sigma_K h_K - \eta_K,\ \ \beta_K \in [0^\circ, 180^\circ]\,. \tag{3.4}$$

$$\eta_K := \frac{\min\{\phi(p), \phi(q)\} - \phi(r)}{h_K}, \tag{3.5}$$

In particular, J is bounded and away from zero independently of h.

(iv) *$\pi : \Gamma_h \to \Gamma$ is a homeomorphism. Consequently, if γ is a connected component of Γ, then $\gamma_h = \{x \in \Gamma_h : \pi(x) \in \gamma\}$ is a simple, closed curve.*

3.1. Discussion of the statement. With Γ and Γ_h as defined in the statement, Theorem 3.1 asserts sufficient conditions under which $\pi : \Gamma_h \to \Gamma$ is a homeomorphism. The statement of the theorem extends also to the case when edges in Γ_h are identified using the function $-\phi$ instead of ϕ. This corresponds to selecting the collection of *negative* edges for parameterizing Γ. Of course, a different collection of angles are required to be acute. If triangles in the vicinity of the curve are all acute angled, the theorem shows that there are two different collections of edges homeomorphic to Γ.

We make three important assumptions on the (family of) background meshes. The first is, expectedly, a sufficiently small mesh size h. For if h is too large, then π may not even be single valued over Γ_h. In §3.2, we provide an explicit upper bound for the required mesh size h_0. The second assumption, quasi-uniformity, is required to control the aspect ratio of triangles as the mesh size is reduced.

Assumption (3.1), which we term the *acute conditioning angle assumption*, is perhaps less intuitive. For once the set Γ_h has been identified, the angles positive edges make with other edges in the background mesh $\mathcal{T}_h$ is irrelevant. Rather, the rationale behind assuming (3.1) is that it provides a means to control the orientation of positive edges with respect to local normals to the curve. We explain this idea below using a simple example.

3.1.1. The acute conditioning angle assumption. To avoid restrictions on the mesh size, assume that Γ is practically straight, see Fig. 3.1. Triangle K shown in the figure is positively cut by Γ, has positive edge e_{ab} and proximal vertex a. Abusing the definition in (3.4), we have $\cos\beta_K = -\eta_K h_K / d(a,c)$ as indicated in the figure (the two definitions coincide if the length of the edge e_{ac} is h_K). The projection of e_{ab} onto Γ has length $d(a,b)\sin(\beta_K - \vartheta_K)$. For π to be injective over e_{ab}, we need to ensure that $0^\circ < \beta_K - \vartheta_K < 180^\circ$. Even though β_K is strictly larger than 90° (as depicted in Fig. 3.1), it can be arbitrarily close to 90°. Therefore, we request that the conditioning angle ϑ_K be smaller than 90°. Consequently, $\beta_K - \vartheta_K > 0^\circ$. The assumptions $\phi(a) \le \phi(b)$ and $\vartheta_K < 90^\circ$ together imply that $\beta_K - \vartheta_K < 180^\circ$.

We refer to [17] for simple examples where π fails to be injective over Γ_h because the conditioning angle fails to be acute. Of course, (3.1) is only a sufficient condition for injectivity. Indeed, a simple way to relax assumption (3.1) is by defining an equivalence relation $\overset{\Gamma}{\simeq}$ over the family of triangulations in which Γ is immersed. Consider

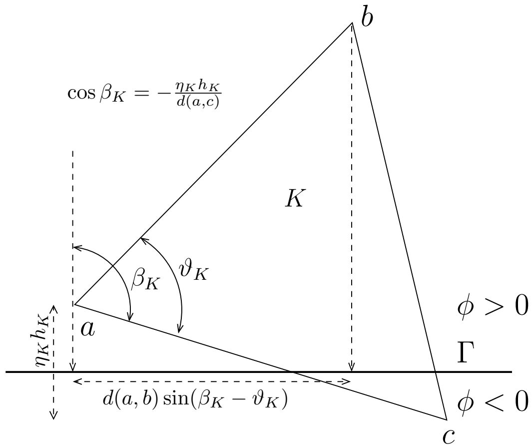

Fig. 3.1: Illustration to explain the rationale behind the acute conditioning angle assumption. Triangle K is positively cut by Γ. Although $\beta_K > 90°$, it can be arbitrarily close to $90°$. Requesting $\vartheta_K < 90°$ ensures that $\beta_K - \vartheta_K > 0°$ and hence that $\pi(e_{ab})$ has non-zero length.

two triangulations $\mathcal{T}_h = (V, C)$ and $\mathcal{T}'_{h'} = (V', C')$. We say $\mathcal{T}_h \overset{\Gamma}{\simeq} \mathcal{T}'_{h'}$ if there is a bijection $\Phi : V \to V'$ such that for any $v \in V$

(i) $(p, q, r) \in C \iff (\Phi(p), \Phi(q), \Phi(r)) \in C'$,
(ii) $\phi(v) \geq 0 \iff \phi(\Phi(v)) \geq 0$,
(iii) $\phi(v) < 0 \iff \phi(\Phi(v)) < 0$,
(iv) $v \in \Gamma_h \Rightarrow \Phi(v) = v$.

The map Φ can be interpreted as a (constrained) perturbation of vertices in $\mathcal{T}_h$ to yield a new mesh $\mathcal{T}'_{h'}$. It is clear from the definition of the equivalence relation that both $\mathcal{T}_h$ and $\mathcal{T}'_{h'}$ have *exactly the same set of positive edges* even though their positively cut triangles can have very different conditioning angles. The key point is that the result of the theorem can be applied to $\mathcal{T}_h$ from merely knowing the existence of a triangulation in its equivalence class that has acute conditioning angle. In light of this observation, the theorem applies even to some families of background meshes that do not satisfy assumption (3.1).

3.1.2. Bound for the Jacobian. Eq.(3.3) provides an estimate for the Jacobian of the parameterization. Inspecting the lower bound in (3.3), which is the critical one, shows that $J \geq \sin(\beta_K - \vartheta_K)$ if $M_K h_K = 0$. This is precisely the Jacobian computed for a line, as in figure 3.1, when the definitions of β_K in (3.4) is replaced by that in the figure. The same interpretation of the lower bound holds when $M_K \neq 0$ but h_K is small. In this case, each positive edge parameterizes a small subset of Γ, which appears essentially straight.

For reasonably large values of $M_K h_K$, the angle β_K in (3.4) can be close to $90°$, even acute. Hence $\beta_K - \vartheta_K$ can be small. In light of this, we mention that a smaller conditioning angle yields a better parameterization, one with J closer to 1.

3.2. An explicit estimate for h_0. By tracking the restrictions on the mesh size in the proof, we can provide an explicit estimate for h_0 in Theorem 3.1. For a C^2-regular boundary Γ, let r_n be the constant given in Theorem 2.2 and let σ and τ

be defined as in (2.2). We require that for each $K \in \mathcal{P}_h$,

$$h_K < r_n, \tag{3.6a}$$

$$M_K h_K < 1, \tag{3.6b}$$

$$\sigma_K C_K^h h_K < \min\left\{\cos\vartheta_K, \sin\frac{\vartheta_K}{2}\right\}, \tag{3.6c}$$

$$C_K^h h < \frac{\tau}{\sigma}. \tag{3.6d}$$

With $M := \max_\Gamma \kappa$ and

$$\theta_c := \inf_h \min_{K\in\mathcal{P}_h} \vartheta_K, \tag{3.7}$$

all conditions in (3.6) are satisfied by selecting

$$h_0 < \min\left\{r_n, \frac{\cos\Theta_c}{M(\sigma+\cos\Theta_c)}, \frac{\sin(\theta_c/2)}{M(\sigma+\sin(\theta_c/2))}, \frac{\tau}{M(\sigma+\tau)}\right\}. \tag{3.8}$$

Since quasi-uniformity implies the lower bound $2\arctan(\sigma/2)$ for angles in $\mathcal{T}_h$ (see [13, chapter 4]), it follows that $\theta_c > 0$ in (3.7). The local restrictions in (3.6) are more useful than (3.8) when considering adaptively refined background meshes.

Such an explicit estimate for the mesh size is useful for two reasons. First, it makes transparent what parameters of the curve and background mesh are relevant. For instance, a smaller mesh size is required when the curve has larger curvatures/small features (dependence on r_n) as well as if the conditioning angle is close to 90° (dependence on ϑ_K). Secondly, these bounds can be computed, at least approximately. As a simple example, consider a circle of radius R immersed in a family of triangulations consisting of all equilateral triangles. For each positively cut triangle K, we have $h_K = h, \vartheta_K = 60°, \sigma_K = \sqrt{3}$ and $M_K = 1/R$. Then, satisfying the three conditions in (3.6) requires $h < h_0 = R/(1+2\sqrt{3}) \simeq 0.224R$. The *a priori* estimate $h_0 = 0.224R$ is a reasonable one because it is comparable to R. Of course, the estimate for h_0 will change with the choice of background meshes.

3.3. Outline of proof. We briefly discuss the outline of the proof of Theorem 3.1. The critical step is showing that π is injective over Γ_h. To this end, we proceed in simple steps by considering the restriction of π over each positive edge, then over pairs of intersecting positive edges and finally over connected components of Γ_h.

We start in §4 by computing bounds for the signed distance function ϕ on Γ_h and for angles between positive edges and local tangents/normals to Γ. By requiring that h be small and invoking assumption (3.1), we show that a positive edge is never parallel to a local normal to Γ (Proposition 5.1). From here, we infer that π is injective over each positive edge in Γ_h (Lemma 5.2). The required bounds for the Jacobian in (3.3) also follow easily from the angle estimates. Part (ii) of the theorem is then a simple consequence of the inverse function theorem.

A logical next step is to show that π is injective over each pair of intersecting positive edges. For this, we examine how positive edges in Γ_h intersect. This is the goal of §6. There, we show that precisely two positive edges intersect at each vertex in Γ_h (Corollary 6.8) and conclude that Γ_h is a collection of simple, closed curves. Additionally, we show that two intersecting positive edges lie on either side of the local normal to Γ (Lemma 6.6). This in particular helps show that π is injective over each pair of intersecting positive edges in Γ_h (Proposition 7.2).

Knowing that (i) π is injective over each pair of intersecting positive edges, (ii) each connected component of Γ_h is a simple, closed curve and (iii) π is continuous over Γ_h, we demonstrate (in Lemma 7.1) that π is a homeomorphism over each connected component of Γ_h. What remains to be shown is that precisely one connected component of Γ_h is mapped to each connected component of Γ. We do this in §8 by illustrating that the collection of positive edges that map to a connected component of Γ is itself a connected set (Lemma 8.1).

3.4. Assumptions and notation for subsequent sections. In all results stated in subsequent sections, we presume that the assumptions in the statement of Theorem 3.1 hold. Additionally, we assume that the mesh size satisfies the conditions in (3.6). In several of the intermediate results this last assumption could be substantially relaxed.

We shall denote the unit normal and unit tangent to Γ at $\xi \in \Gamma$ by $\hat{N}(\xi)$ and $\hat{T}(\xi)$ respectively. We assume an orientation for Γ such that $\hat{N}$ is parallel to $\nabla\phi$ and that $\{\hat{T}, \hat{N}\}$ constitutes a right-handed basis for $\mathbb{R}^2$ at any point on the curve. Given distinct points $a, b \in \mathbb{R}^2$, we denote the unit vector pointing from a to b by $\hat{U}_{ab}$ and define $\hat{U}_{ab}^{\perp}$ such that $\{\hat{U}_{ab}, \hat{U}_{ab}^{\perp}\}$ is a right-handed basis.

4. Distance and angle estimates. In this section, we compute bounds for the signed distance function ϕ on Γ_h and estimates useful in bounding the angle between positive edges and local normals to Γ.

PROPOSITION 4.1. *Let $K = (a, b, c) \in \mathcal{P}_h$. Then*

(i) $\overline{K} \cap \Gamma \neq \emptyset$. *In particular,* $\phi(c) < 0 \Rightarrow e_{bc} \cap \Gamma \neq \emptyset, e_{ac} \cap \Gamma \neq \emptyset$,
(ii) $|\phi| \leq h_K$ *on* $\overline{K}$,
(iii) η_K *defined in* (3.5) *satisfies* $0 < \eta_K \leq 1$,
(iv) β_K *given by* (3.4) *is well defined and* $\beta_K > \vartheta_K$.

Proof. We only show (iv) and the upper bound in (iii), since the others follow directly from the definitions. To this end, assume that $\phi(c) < 0$ and $\phi(a), \phi(b) \geq 0$, and consider any $\xi \in e_{ac} \cap \Gamma$. From the definition of η_K in (3.5), we have

$$\eta_K h_K \leq \phi(a) - \phi(c) \leq d(a, \Gamma) + d(c, \Gamma) \leq d(a, \xi) + d(c, \xi) \leq h_K,$$

which shows that $\eta_K \leq 1$. To show that β_K is well defined, we check that $\cos\beta_K \in [-1, 1]$ using $\eta_K \leq 1$, 3.6b and (3.6c):

$$-1 \leq -\eta_K \leq \cos\beta_K = \sigma_K C_K^h h_K - \eta_K \leq \sigma_K C_K^h h_K < \cos\vartheta_K, \tag{4.1}$$

from where it also follows that $\beta_K > \vartheta_K$. $\square$

For any $K \in \mathcal{P}_h$, part (ii) of the above proposition and $h_K < h < r_n$ (from (3.6a)) show that $\overline{K} \subset B(\Gamma, r_n)$. Then Theorem 2.2 shows that π is C^1 and in particular, well defined over $\overline{K}$. Furthermore, since any positive edge in Γ_h is an edge of some triangle in $\mathcal{P}_h$, we get that $\Gamma_h \subset B(\Gamma, r_n)$ and hence that π is well defined and continuous on Γ_h. We shall frequently use these consequences of Proposition 4.1 in the rest of the proof, often without explicitly referring to the proposition.

The following corollary is useful when estimating ϕ and $\nabla\phi$ in positively cut triangles knowing just the values at vertices of the triangle.

COROLLARY 4.2 (of Proposition 2.3). *Let $K \in \mathcal{P}_h$ and $x, y \in \overline{K}$. Then,*

$$\left|\phi(y) - (y - \pi(x)) \cdot \hat{N}(\pi(x))\right| \leq \frac{1}{2} C_K^h d(x, y)^2, \tag{4.2a}$$

$$\textit{and} \quad |\nabla\phi(y) - \nabla\phi(x)| \leq C_K^h d(x, y). \tag{4.2b}$$

Proof. Let $L_{xy} \subset \overline{K}$ be the closed line segment joining x and y. We have

$$\max_{L_{xy}} \kappa \circ \pi \le \max_{\overline{K}} \kappa \circ \pi \le \max_{B(K,h_K)\cap\Gamma} \kappa = M_K \tag{4.3}$$

and $|\phi| \le h_K$ on L_{xy}. Since $M_K h_K < 1$ by (3.6b), Proposition 2.3 implies the bound

$$\left|\hat{U}_{xy} \cdot \nabla\nabla\phi(z) \cdot \hat{U}_{xy}\right| \le \frac{\kappa(\pi(z))}{1 - |\phi(z)|\,\kappa(\pi(z))} \le \frac{M_K}{1 - M_K h_K} = C_h^K \quad \forall z \in L_{xy}. \tag{4.4}$$

From Taylor's theorem, we have

$$\begin{aligned} |\phi(y) - \phi(x) - \nabla\phi(x)\cdot(y-x)| &\le \frac{d(x,y)^2}{2} \max_{L_{xy}} \left|\hat{U}_{xy} \cdot \nabla\nabla\phi \cdot \hat{U}_{xy}\right|, \\ |\nabla\phi(y) - \nabla\phi(x)| &\le d(x,y) \max_{L_{xy}} \left|\hat{U}_{xy} \cdot \nabla\nabla\phi \cdot \hat{U}_{xy}\right| \end{aligned} \tag{4.5}$$

Using (4.4) and $x = \pi(x) + \phi(x)\hat{N}(\pi(x))$ (Theorem 2.2) in (4.5) yields (4.2). ◻

The essential angle estimate we will need is for the angle between positive edges and local normals to Γ. This is computed later in Proposition 5.1 using the angle estimates computed below in Propositions 4.3 and 4.4.

PROPOSITION 4.3. *Let $K = (a,b,c) \in \mathcal{P}_h$ have positive edge e_{ab}. Then*

$$\hat{N}(\pi(x)) \cdot \hat{U}_{yc} \le \cos\beta_K \quad \forall x, y \in e_{ab}. \tag{4.6}$$

Proof. Let $\hat{n}_x = \hat{N}(\pi(x))$. From corollary 4.2, we have

$$\phi(i) \le (i - \pi(x)) \cdot \hat{n}_x + \frac{1}{2} C_K^h h_K^2 \quad \text{for } i = a, b, \tag{4.7a}$$

$$\text{and} \quad \phi(c) \ge (c - \pi(x)) \cdot \hat{n}_x - \frac{1}{2} C_K^h h_K^2. \tag{4.7b}$$

By definition of η_K in (3.5), we know

$$\phi(i) - \phi(c) \ge \eta_K h_K \quad \text{for } i = a, b. \tag{4.8}$$

Using (4.7) in (4.8), we get

$$(c - i) \cdot \hat{n}_x \le C_K^h h_K^2 - \eta_K h_K \quad \text{for } i = a, b. \tag{4.9}$$

Since $y \in e_{ab}$, y is a convex combination of a and b. Therefore (4.9) implies that

$$(c - y) \cdot \hat{n}_x \le C_K^h h_K^2 - \eta_K h_K. \tag{4.10}$$

Dividing (4.10) by $d(c,y)$ and noting that $\rho_K < d(c,y) \le h_K$, we get

$$\hat{U}_{yc} \cdot \hat{n}_x \le C_K^h h_K \frac{h_K}{\rho_K} - \eta_K \frac{h_K}{h_K} = \sigma_K C_K^h h_K - \eta_K = \cos\beta_K, \tag{4.11}$$

which is the required inequality. ◻

PROPOSITION 4.4. *Let $K = (a,b,c) \in \mathcal{P}_h$ have positive edge e_{ab} and proximal vertex a. Then*

$$\hat{N}(\pi(a)) \cdot \hat{U}_{ab} \ge -\frac{1}{2} C_K^h h_K. \tag{4.12}$$

Proof. Since a is the proximal vertex of K, $\phi(a) \leq \phi(b)$. Then, using Theorem 2.2, we get

$$\phi(b) \geq \phi(a) = (a - \pi(a)) \cdot \hat{N}(\pi(a)). \tag{4.13}$$

From Corollary 4.2, we also have

$$\phi(b) \leq (b - \pi(a)) \cdot \hat{N}(\pi(a)) + \frac{1}{2} C_K^h d(a,b)^2. \tag{4.14}$$

Subtracting (4.13) from (4.14), we get

$$(b - a) \cdot \hat{N}(\pi(a)) \geq -\frac{1}{2} C_K^h d(a,b)^2. \tag{4.15}$$

Dividing (4.15) by $d(a,b)$ and using $d(a,b) \leq h_K$ yields

$$\hat{U}_{ab} \cdot \hat{N}(\pi(a)) \geq -\frac{1}{2} C_K^h d(a,b) \geq -\frac{1}{2} C_K^h h_K, \tag{4.16}$$

which is the required inequality. ☐

Part (ii) of Proposition 4.1 implies the lower bound $\phi \geq -h$ on Γ_h. This can be improved, since $\phi \geq 0$ at each vertex in Γ_h. We will use the h^2 scaling shown below later in §8.

Corollary 4.5 (of Proposition 4.4). *Let $K = (a,b,c) \in \mathcal{P}_h$ have positive edge e_{ab}. Then*

$$\phi(x) > -C_K^h h_K^2 \quad \forall x \in e_{ab}. \tag{4.17}$$

Proof. Let $x \neq b$, since $\phi(b) \geq 0$. From Corollary 4.2, we have

$$\begin{aligned}
\phi(x) &\geq (x - \pi(a)) \cdot \hat{N}(\pi(a)) - \frac{1}{2} C_K^h d(a,x)^2, \\
&= \phi(a) + (x - a) \cdot \hat{N}(\pi(a)) - \frac{1}{2} C_K^h d(a,x)^2 \quad \Big(\text{using } \pi(a) = a - \phi(a)\hat{N}(\pi(a))\Big) \\
&\geq \phi(a) - \frac{1}{2} d(x,a) C_K^h h_K - \frac{1}{2} C_K^h d(a,x)^2, \qquad \text{(Proposition 4.4)} \\
&> -C_K^h h_K^2. \qquad (\phi(a) \geq 0, d(a,x) < h_K) \qquad ☐
\end{aligned}$$

5. Injectivity on each positive edge. We can now estimate the angle between a positive edge and the local normal to Γ. The idea behind the calculation is essentially the one illustrated in Fig. 3.1.

Proposition 5.1. *Let $K = (a,b,c) \in \mathcal{P}_h$ have positive edge e_{ab} and proximal vertex a. Then*

$$-\frac{3}{2} C_K^h h_K \leq \hat{N}(\pi(x)) \cdot \hat{U}_{ab} \leq \cos(\beta_K - \vartheta_K) \quad \forall x \in e_{ab}. \tag{5.1}$$

In particular, $|\hat{N}(\pi(x)) \cdot \hat{U}_{ab}| < 1$ and $|\hat{T}(\pi(x)) \cdot \hat{U}_{ab}| > 0$.

Proof. We first obtain the lower bound in (5.1) by using the bound for $\hat{N}(\pi(a))\cdot\hat{U}_{ab}$ derived in Proposition 4.4. We have

$$\begin{aligned}
\hat{N}(\pi(x))\cdot\hat{U}_{ab} &= \hat{N}(\pi(a))\cdot\hat{U}_{ab} + \Big(\hat{N}(\pi(x)) - \hat{N}(\pi(a))\Big)\cdot\hat{U}_{ab}, \\
&\geq -\frac{1}{2}C_K^h h_K - \Big|\hat{N}(\pi(x)) - \hat{N}(\pi(a))\Big|, \qquad \text{(Proposition 4.4)} \\
&= -\frac{1}{2}C_K^h h_K - \left|\nabla\phi(x) - \nabla\phi(a)\right|, \\
&\geq -\frac{1}{2}C_K^h h_K - C_K^h h_K, \qquad \text{(corollary 4.2)}
\end{aligned}$$

which proves the lower bound.
To derive the upper bound, we make use of the inequality

$$\arccos(\hat{u}\cdot\hat{v}) \leq \arccos(\hat{u}\cdot\hat{w}) + \arccos(\hat{v}\cdot\hat{w}), \tag{5.2}$$

for any three unit vectors $\hat{u},\hat{v},\hat{w}$ in $\mathbb{R}^2$, with $\arccos\colon [-1,1]\to[0,\pi]$. Setting $\hat{u} = \hat{N}(\pi(x))$, $\hat{v} = \hat{U}_{ac}$ and $\hat{w} = \hat{U}_{ab}$ in (5.2), we get

$$\arccos(\hat{N}(\pi(x))\cdot\hat{U}_{ab}) \geq \arccos(\hat{N}(\pi(x))\cdot\hat{U}_{ac}) - \arccos(\hat{U}_{ac}\cdot\hat{U}_{ab}). \tag{5.3}$$

From Proposition 4.3, we know $\hat{N}(\pi(x))\cdot\hat{U}_{ac} \leq \cos\beta_K$. Since a is the proximal vertex in K, we have $\hat{U}_{ac}\cdot\hat{U}_{ab} = \cos\vartheta_K$. The upper bound in (5.1) follows.

Finally, to demonstrate that $\Big|\hat{N}(\pi(x))\cdot\hat{U}_{ab}\Big| < 1$, it suffices to show that $\frac{3}{2}C_K^h h_K$ and $\cos(\beta_K - \vartheta_K)$ are both smaller than 1. The latter follows from part (iv) of Proposition 4.1. For the former, noting that $\sigma_K \geq \sqrt{3}$ in (3.6c) yields $(3/2)C_K^h h_K \leq \sigma_K C_K^h h_K < \sin\vartheta_K/2 < 1$. □

That a positive edge is never parallel to a local normal to Γ immediately implies injectivity of π over each positive edge, as shown next.

Lemma 5.2. *The restriction of π to each positive edge in Γ_h is injective.*

Proof. Let $(a,b,c)\in\mathcal{P}_h$ have positive edge e_{ab} and proximal vertex a. We proceed by contradiction. Suppose that $x,y\in e_{ab}$ are distinct points such that $\pi(x)=\pi(y)$. From Theorem 2.2 and $\pi(x)=\pi(y)$, we have

$$x = \pi(x) + \phi(x)\hat{N}(\pi(x)), \tag{5.4a}$$

$$y = \pi(y) + \phi(y)\hat{N}(\pi(y)) = \pi(x) + \phi(y)\hat{N}(\pi(x)). \tag{5.4b}$$

Noting $x\neq y$ in (5.4) implies that $\phi(x)\neq\phi(y)$. Therefore, subtracting (5.4b) from (5.4a) yields

$$\hat{N}(\pi(x)) = \frac{x-y}{\phi(x)-\phi(y)}. \tag{5.5}$$

By definition of $x,y\in e_{ab}$, $x-y$ is a vector parallel to $\hat{U}_{ab}$. Therefore (5.5) in fact shows that $|\hat{N}(\pi(x))\cdot\hat{U}_{ab}| = 1$, contradicting Proposition 5.1. □

As noted in §4, continuity of π on each positive edge follows from part (ii) Proposition 4.1. The continuity of the inverse is a consequence of Lemma 5.2 and the following result in basic topology, which we use here and later in §7.

Theorem 5.3 ([2, chapter 3]). *A one-one, onto and continuous function from a compact space to a Hausdorff space is a homeomorphism.*

Corollary 5.4 (of Lemma 5.2). *Let e be a positive edge in Γ_h. Then $\pi : e \to \pi(e)$ is a homeomorphism.*

Proof. From part (ii) of Proposition 4.1 and Lemma 5.2, we know that π is continuous and injective on e. The corollary then follows from Theorem 5.3. □

The above corollary is an important step in proving part (iv) of Theorem 3.1. Extending such a result to the entire collection of positive edges is the objective of subsequent sections. With the angle estimate in Proposition 5.1, we can demonstrate the bounds for the Jacobian in (3.3). Noting that the Jacobian is non-zero and invoking the inverse function theorem then proves part (ii) of Theorem 3.1.

Lemma 5.5. *Let $K = (a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab}. Then π is C^1 over $\boldsymbol{ri}\,(e_{ab})$ and*

$$0 < \frac{\sin(\beta_K - \vartheta_K)}{1 + M_K h_K} \le \left|\nabla\pi(x) \cdot \hat{U}_{ab}\right| \le \frac{1}{1 - M_K h_K} \le \frac{5}{3} \quad \forall x \in e_{ab}. \tag{5.6}$$

Proof. From part (ii) of Proposition 4.1 and (3.6a), we know $e_{ab} \subset B(\Gamma, r_n)$. Then Theorem 2.2 shows that π is C^1 over $\mathbf{ri}\,(e_{ab})$.

Consider any $x \in \mathbf{ri}\,(e_{ab})$. Since $|\phi(x)| \le h_K$ (Proposition 4.1),

$$\kappa(\pi(x)) \le \max_{B(x,h_K)\cap\Gamma} \kappa \le \max_{B(K,h_K)\cap\Gamma} \kappa = M_K. \tag{5.7}$$

Therefore, $|\phi(x)\kappa(\pi(x))| \le M_K h_K$ which is smaller than 1 by (3.6b). Then from Proposition 2.3, we get

$$J(x) := \left|\nabla\pi(x) \cdot \hat{U}_{ab}\right| = \frac{|\hat{U}_{ab} \cdot \hat{T}(\pi(x))|}{|1 - \phi(x)\kappa_s(\pi(x))|}, \tag{5.8}$$

where κ_s is the signed curvature of Γ (and $\kappa = |\kappa_s|$). Noting that $|\phi(x)\kappa_s(\pi(x))| \le M_K h_K$ in (5.8), we get

$$1 - M_K h_K \le |1 - \phi(x)\kappa_s(\pi(x))| \le 1 + M_K h_K. \tag{5.9}$$

From Proposition 5.1, we have

$$|\sin(\beta_K - \vartheta_K)| \le \left|\hat{T}(\pi(x)) \cdot \hat{U}_{ab}\right| \le 1. \tag{5.10}$$

Note however from part (iv) of Proposition 4.1 that $\beta_K > \vartheta_K \Rightarrow |\sin(\beta_K - \vartheta_K)| = \sin(\beta_K - \vartheta_K)$. Then using (5.9) and (5.10) in (5.8) yields the lower and upper bounds for $|J(x)|$ in (5.6).

It remains to show that these bounds are meaningful, i.e., the lower bound is positive and the upper bound is not arbitrarily large. The former is a consequence of $\beta_K > \vartheta_K$ (from Proposition 4.1). Using then (3.6b) and that $\sigma_K \ge \sqrt{3}$ in (3.6c), it follows that $\sigma_K C_K^h h_K < \sin\frac{\vartheta_K}{2} \le 1$, and hence $M_K h_K < (1 + \sqrt{3})^{-1} < 2/5$, which renders an h_K-independent bound for the upper bound in (5.6). □

6. The set Γ_h. An essential step in showing that π is injective over Γ_h is understanding how positive edges intersect. The goal of this section is to demonstrate that Γ_h is a union of simple, closed curves (Lemma 6.9). We achieve this by considering how many positive edges intersect at each vertex in Γ_h. In Lemma 6.6, we show that this number is precisely two. In this section and the remainder of the article, we will

use the sign function defined over $\mathbb{R}$ as $\operatorname{sgn}(x) = x/|x|$ if $x \neq 0$ and $\operatorname{sgn}(x) = 0$ if $x = 0$.

PROPOSITION 6.1. *Let $(a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab} and proximal vertex a. Then*

$$\operatorname{sgn}(\hat{T}(\pi(a)) \cdot \hat{U}_{ab}) = \operatorname{sgn}(\hat{T}(\pi(a)) \cdot \hat{U}_{ac}) \neq 0, \tag{6.1a}$$

$$\hat{N}(\pi(a)) \cdot \hat{U}_{ac} < \hat{N}(\pi(a)) \cdot \hat{U}_{ab}. \tag{6.1b}$$

Proof. For convenience, let $\hat{t} = \hat{T}(\pi(a))$ and $\hat{n} = \hat{N}(\pi(a))$. Let $\alpha_b, \alpha_c \in [0°, 360°)$ denote the angles from $\hat{n}$ to $\hat{U}_{ab}$ and $\hat{U}_{ac}$ respectively measured in the clockwise sense so that

$$\hat{U}_{ai} = \cos\alpha_i\,\hat{n} + \sin\alpha_i\,\hat{t} \quad \text{for } i = b, c. \tag{6.2}$$

From (6.2) and the assumption that a is the proximal vertex in K, note that

$$\cos\vartheta_K = \hat{U}_{ab} \cdot \hat{U}_{ac} = \cos\alpha_b \cos\alpha_c + \sin\alpha_b \sin\alpha_c = \cos(\alpha_c - \alpha_b). \tag{6.3}$$

First we prove (6.1a). Since Proposition 5.1 shows $\hat{t} \cdot \hat{U}_{ab} \neq 0$, without loss of generality assume that $\hat{t} \cdot \hat{U}_{ab} > 0 \Rightarrow \alpha_b \in (0°, 180°)$. The upper bound can be improved by invoking Proposition 4.4, (3.6c) and $\sigma_K \geq \sqrt{3}$:

$$\cos\alpha_b = \hat{n} \cdot \hat{U}_{ab} \geq -\frac{1}{2} C_K^h h_K \geq -\sigma_K C_K^h h_K > -\cos\vartheta_K \Rightarrow \alpha_b < 180° - \vartheta_K. \tag{6.4}$$

Suppose then that $\hat{t} \cdot \hat{U}_{ac} \leq 0$, i.e., $\alpha_c \geq 180°$. From Propositions 4.1 and 4.3, we have $\alpha_c \leq 360° - \beta_K < 360° - \vartheta_K$. In conjunction with (6.4), this shows $\vartheta_K \leq (\alpha_c - 180°) + \vartheta_K < \alpha_c - \alpha_b < 360° - \vartheta_K$ which clearly contradicts (6.3). Therefore $\hat{t} \cdot \hat{U}_{ab} > 0 \Rightarrow \hat{t} \cdot \hat{U}_{ac} > 0$ as well. The case $\hat{t} \cdot \hat{U}_{ab} < 0$ can be argued similarly.

Next we show (6.1b). Following (6.1a), without loss of generality assume that $\hat{t} \cdot \hat{U}_{ab}$ and $\hat{t} \cdot \hat{U}_{ac}$ are both positive. Consequently, $\alpha_b, \alpha_c \in (0°, 180°)$. We proceed by contradiction. Suppose that $\hat{n} \cdot \hat{U}_{ab} \leq \hat{n} \cdot \hat{U}_{ac} \Rightarrow \alpha_c \leq \alpha_b$. Then, noting that $\cos\beta_K < \sigma_K C_K^h h_K$ (from (3.4) and Proposition 4.1 part (iii)), $\cos\alpha_c \leq \cos\beta_K$ (Proposition 4.3) and $\cos\alpha_b \geq -\sigma_K C_K^h h_K$ (Proposition 4.4, $\sigma_K \geq \sqrt{3}$), we get

$$90° - \arcsin(\sigma_K C_K^h h_K) < \beta_K \leq \alpha_c \leq \alpha_b \leq 90° + \arcsin(\sigma_K C_K^h h_K), \tag{6.5}$$

where $\arcsin\colon [-1, 1] \to [-\pi/2, \pi/2]$. Together with (3.6c), this implies that $\alpha_b - \alpha_c < 2\arcsin(\sigma_K C_K^h h_K) < 2 \times \vartheta_K/2 = \vartheta_K$, which contradicts (6.3), and hence $\hat{n} \cdot \hat{U}_{ab} > \hat{n} \cdot \hat{U}_{ac}$. Again, the case in which both terms in (6.1a) are negative is handled similarly. ∎

PROPOSITION 6.2. *Let $(a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab}. Then*

$$\operatorname{sgn}(\hat{T}(\pi(x)) \cdot \hat{U}_{ab}) = \operatorname{sgn}(\hat{U}_{ca} \cdot \hat{U}_{ab}^{\perp}) = \operatorname{sgn}(\hat{U}_{cb} \cdot \hat{U}_{ab}^{\perp}) \quad \forall x \in e_{ab}. \tag{6.6}$$

Proof. Notice first that since

$$d(c, a)\hat{U}_{ca} = d(c, b)\hat{U}_{cb} + d(b, a)\hat{U}_{ba}, \tag{6.7}$$

it follows that $\operatorname{sgn}(\hat{U}_{ca} \cdot \hat{U}_{ab}^{\perp}) = \operatorname{sgn}(\hat{U}_{cb} \cdot \hat{U}_{ab}^{\perp})$, after taking the inner product on both sides with $\hat{U}_{ab}^{\perp}$. Without loss of generality then, assume that the proximal vertex in

triangle (a,b,c) is the vertex a. For convenience, let $\alpha_i = \arccos(\hat{N}(\pi(a)) \cdot \hat{U}_{ai})$ for $i = b,c$. From Proposition 6.1, we know $\mathrm{sgn}(\hat{U}_{ab} \cdot \hat{T}(\pi(a))) = \mathrm{sgn}(\hat{T}(\pi(a)) \cdot \hat{U}_{ac}) := \iota$. From the definition of α_b, α_c and ι, we have

$$\hat{U}_{ai} = \cos\alpha_i \ \hat{n} + \iota \sin\alpha_i \ \hat{t} \quad \text{for } i = b,c, \tag{6.8a}$$

$$\hat{U}_{ab}^{\perp} = \iota \sin\alpha_b \ \hat{n} - \cos\alpha_b \ \hat{t}, \tag{6.8b}$$

where we have again set $\hat{t} = \hat{T}(\pi(a))$ and $\hat{n} = \hat{N}(\pi(a))$. Noting that $0^\circ < \alpha_b < 180^\circ$ from Proposition 5.1 and $\alpha_b < \alpha_c$ from Proposition 6.1, we get $0^\circ < \alpha_c - \alpha_b < 180^\circ$. Then, using (6.8), we have the following calculation:

$$\mathrm{sgn}(\hat{U}_{ca} \cdot \hat{U}_{ab}^{\perp}) = \mathrm{sgn}(\iota \sin(\alpha_c - \alpha_b)) = \iota = \mathrm{sgn}(\hat{t} \cdot \hat{U}_{ab}), \tag{6.9}$$

which proves (6.6) for $x = a$. This in fact implies (6.6) for every $x \in e_{ab}$. For if we suppose otherwise, then by continuity of the mapping $\hat{U}_{ab} \cdot \left(\hat{T} \circ \pi\right) : e_{ab} \to \mathbb{R}$, there would exist $y \in e_{ab}$ such that $\hat{U}_{ab} \cdot \hat{T}(\pi(y)) = 0$, contradicting Proposition 5.1. ◻

The definition of a positive edge in §2 does not forbid a positive edge to belong to two positively cut triangles. As claimed in part (i) of Theorem 3.1, this cannot be the case under the hypotheses of the theorem.

Lemma 6.3. *Each positive edge in $\mathcal{T}_h$ is a positive edge of precisely one triangle positively cut by Γ.*

Proof. Let e_{ab} be a positive edge in Γ_h. By definition, we can find $K = (a,b,c) \in \mathcal{P}_h$ for which e_{ab} is a positive edge. Suppose that there exists $\tilde{K} = (a,b,d) \in \mathcal{P}_h$ different from K that also has positive edge e_{ab}. Then, applying Proposition 6.2 to triangles K and $\tilde{K}$, we get

$$\mathrm{sgn}(\hat{U}_{ab}^{\perp} \cdot \hat{U}_{ca}) = \mathrm{sgn}(\hat{U}_{ab}^{\perp} \cdot \hat{U}_{da}), \tag{6.10}$$

because both equal $\mathrm{sgn}(\hat{U}_{ab} \cdot \hat{T}(\pi(a)))$. But (6.10) implies that $K \cap \tilde{K} \neq \emptyset$. This is a contradiction since K and $\tilde{K}$ are non-overlapping open sets. ◻

The following proposition is useful for subsequent arguments in this section.

Proposition 6.4. *Let e_{pq} be an edge in $\mathcal{T}_h$ such that $\phi(p) \geq 0$ and $\phi(q) < 0$. Then e_{pq} is an edge of two distinct triangles in $\mathcal{T}_h$.*

Proof. Let ω_h be the domain triangulated by $\mathcal{T}_h$. To prove the lemma, it suffices to find a non-empty open ball centered at any point in e_{pq} and contained in ω_h. To this end, note that since ϕ is continuous on e_{pq} and has opposite signs at vertices p and q, we can find $\xi \in \Gamma \cap e_{pq}$. Since Γ is assumed to be immersed in $\mathcal{T}_h$, we know that $\Gamma \subset \mathrm{int}(\omega_h)$. Therefore, there exists $\varepsilon > 0$ such that $B(\xi, \varepsilon) \subset \mathrm{int}(\omega_h)$, which is the required ball. ◻

The following lemma is the essential step in showing that connected components of Γ_h are closed curves.

Lemma 6.5. *At least two positive edges intersect at each vertex in Γ_h.*

Proof. Let a be any vertex in Γ_h. Since Γ_h is the union of positive edges in $\mathcal{T}_h$, it follows that a is a vertex of at least one positive edge. Suppose that a is a vertex of just one positive edge, say e_{ab_0}. Then, we can find a triangle $(a, b_0, b_1) \in \mathcal{P}_h$ that has positive edge e_{ab_0}. Since $\phi(a) \geq 0$ and $\phi(b_1) < 0$, applying Proposition 6.4 to edge e_{ab_1} shows that there exists $(a, b_1, b_2) \in \mathcal{T}_h$ different from (a, b_0, b_1). Since e_{ab_2} is not a positive edge, we know $\phi(b_2) < 0$. Repeating this step, we find distinct vertices $b_1, b_2, \ldots b_n$ such that $(a, b_i, b_{(i+1)}) \in \mathcal{T}_h$ for $i = 0$ to $n-1$, $\phi(b_i) < 0$ for $i = 1$ to $n-1$ and terminate when b_n coincides with b_0. That n is finite follows from the assumption

of finite number of vertices in $\mathcal{T}_h$. In particular, we have shown that (a, b_0, b_1) and (a, b_{n-1}, b_0) are distinct triangles in $\mathcal{T}_h$ that are both positively cut by Γ and have positive edge e_{ab_0}. This contradicts Lemma 6.3. □

We now show that precisely two distinct positive edges intersect at each vertex in Γ_h. We construct this result essentially in the next lemma.

LEMMA 6.6. *If e_{ap} and e_{aq} are distinct positive edges in $\mathcal{T}_h$, then*

$$\operatorname{sgn}(\hat{U}_{ap} \cdot \hat{T}(\pi(a))) = -\operatorname{sgn}(\hat{U}_{aq} \cdot \hat{T}(\pi(a)) \neq 0. \tag{6.11}$$

To prove the lemma, we will use the following corollary of Proposition 6.2. Note the difference with Proposition 6.1: below a does not need to be the proximal vertex.

COROLLARY 6.7 (of Proposition 6.2). *Let $(a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab} and denote $\hat{t} = \hat{T}(\pi(a))$ and $\hat{n} = \hat{N}(\pi(a))$. Then*

$$\operatorname{sgn}(\hat{t} \cdot \hat{U}_{ab}) = \operatorname{sgn}(\hat{t} \cdot \hat{U}_{ac}) \Rightarrow \hat{n} \cdot \hat{U}_{ab} > \hat{n} \cdot \hat{U}_{ac}. \tag{6.12}$$

Proof. Let $\operatorname{sgn}(\hat{t} \cdot \hat{U}_{ab}) = \operatorname{sgn}(\hat{t} \cdot \hat{U}_{ac}) = \iota$ and $\alpha_i = \arccos(\hat{n} \cdot \hat{U}_{ai})$ for $i = b, c$. Since edges e_{ab} and e_{ac} in triangle (a, b, c) cannot be parallel, we know $\hat{n} \cdot \hat{U}_{ab} \neq \hat{n} \cdot \hat{U}_{ac}$. Then, we have that

$$\hat{U}_{ca} \cdot \hat{U}_{ab}^{\perp} = -\left(\cos\alpha_c\, \hat{n} + \iota\, \sin\alpha_c\, \hat{t}\right) \cdot \left(\iota \sin\alpha_b\, \hat{n} - \cos\alpha_b\, \hat{t}\right) = \iota \sin(\alpha_c - \alpha_b),$$

and from Proposition 6.2, we get

$$\iota = \operatorname{sgn}(\hat{t} \cdot \hat{U}_{ab}) = \operatorname{sgn}(\hat{U}_{ca} \cdot \hat{U}_{ab}^{\perp}) = \operatorname{sgn}(\iota \sin(\alpha_c - \alpha_b)) = \iota\ \operatorname{sgn}(\sin(\alpha_c - \alpha_b)). \tag{6.13}$$

Since $\iota \neq 0$ from Proposition 5.1, we conclude that $\operatorname{sgn}(\sin(\alpha_c - \alpha_b)) = 1$ and hence that $\alpha_c > \alpha_b$. □

Proof. [Proof of Lemma 6.6] We proceed by contradiction. Let $\hat{t} = \hat{T}(\pi(a))$ and $\hat{n} = \hat{N}(\pi(a))$. Proposition 5.1 shows that neither term in (6.11) equals zero. Therefore, without loss of generality, suppose that

$$\operatorname{sgn}(\hat{t} \cdot \hat{U}_{ap}) = \operatorname{sgn}(\hat{t} \cdot \hat{U}_{aq}) = 1. \tag{6.14}$$

Since e_{ap} and e_{aq} are distinct edges, (6.14) implies that $\hat{n} \cdot \hat{U}_{ap} \neq \hat{n} \cdot \hat{U}_{aq}$. Therefore, without loss of generality, we assume that

$$\hat{n} \cdot \hat{U}_{ap} > \hat{n} \cdot \hat{U}_{aq}. \tag{6.15}$$

Let $\{p_1, \ldots, p_n\}$ be a clockwise enumeration of all vertices in $\mathcal{T}_h$ such that e_{ap_i} is an edge in $\mathcal{T}_h$ for each $i = 1$ to n and $p_1 = p$. Let $m \leq n$ be such that $q = p_m$. Without loss of generality, we assume that e_{ap_i} is not a positive edge for $i = 2$ to $m-1$. Denote by $\alpha_i \in [0°, 360°)$, the angle between $\hat{n}$ and $\hat{U}_{ap_i}$ measured in the clockwise sense. From (6.14) and (6.15), we get that $0° < \alpha_1 < \alpha_m < 180°$. Using the clockwise ordering of vertices, this implies that

$$0° < \alpha_1 < \alpha_2 < \ldots < \alpha_m < 180°. \tag{6.16}$$

Arguing by contradiction, we now show that $(a, p_1, p_2) \in \mathcal{T}_h$ and is positively cut. Suppose that (a, p_1, p_2) is not positively cut. Then since e_{ap_1} is a positive edge,

$(a, p_n, p_1) \in \mathcal{T}_h$ and is positively cut. Note that the interior angle at a in (a, p_n, p_1), namely the angle between edges e_{ap_n} and e_{ap_1} measured in the clockwise sense, has to be smaller than 180°. Therefore, either $\alpha_n < \alpha_1$ or $\alpha_n - \alpha_1 > 180°$. In either case, we have

$$\hat{U}_{p_n a} \cdot \hat{U}^{\perp}_{ap_1} = -(\cos\alpha_n\, \hat{n} + \sin\alpha_n\, \hat{t}) \cdot (\sin\alpha_1\, \hat{n} - \cos\alpha_1\, \hat{t}) = \sin(\alpha_n - \alpha_1) < 0. \quad (6.17)$$

Using Proposition 6.2 in (a, p_1, p_n), (6.14) and (6.17), we get

$$1 = \mathrm{sgn}(\hat{U}_{ap_1} \cdot \hat{t}) = \mathrm{sgn}(\hat{U}_{p_n a} \cdot \hat{U}^{\perp}_{ap_1}) = -1,$$

which is a contradiction. Hence, we conclude that $(a, p_1, p_2) \in \mathcal{T}_h$ and is positively cut.

Triangle (a, p_1, p_2) being positively cut with positive edge e_{ap_1} implies $\phi(p_2) < 0$. Then Proposition 6.4 shows that $(a, p_2, p_3) \in \mathcal{T}_h$. If $m \neq 3$, then $\phi(p_3) < 0$ since e_{ap_3} is not a positive edge. Repeating this step, we show that $(a, p_i, p_{(i+1)}) \in \mathcal{T}_h$ for $i = 1$ to $m-1$ and that $\phi(p_i) < 0$ for $i = 2$ to $m-1$. In particular, we get that $(a, p_{m-1}, p_m) \in \mathcal{T}_h$ and is positively cut. This contradicts corollary 6.7 because (6.16) shows that $\mathrm{sgn}(\hat{t} \cdot \hat{U}_{ap_{m-1}}) = \mathrm{sgn}(\hat{t} \cdot \hat{U}_{ap_m})$ and $\hat{n} \cdot \hat{U}_{ap_{m-1}} > \hat{n} \cdot \hat{U}_{ap_m}$.
An identical argument with an anti-clockwise ordering of vertices applies to the case when $\hat{t} \cdot \hat{U}_{ap}$ and $\hat{t} \cdot \hat{U}_{aq}$ are negative. □
The following corollary is an immediate consequence of Lemma 6.6.

Corollary 6.8 (of Lemma 6.6). *Precisely two distinct positive edges intersect at each vertex in* Γ_h.

Lemma 6.9. *Let* γ_h *be a connected component of* Γ_h. *Then* γ_h *is a simple, closed curve that can be represented as as*

$$\gamma_h = \bigcup_{i=0}^{n} e_{v_i v_{(i+1) mod\,(n)}}, \quad (6.18)$$

where $v_0, \ldots, v_n$ *are all the distinct vertices in* γ_h *and* $2 \leq n < \infty$.

Proof. We will only prove (6.18). That γ_h is a simple and closed curve follows immediately from such a representation.

Denote the number of vertices in γ_h by $n+1$ for some integer n. Since γ_h is non-empty, it contains at least one positive edge, say $e_{v_0 v_1}$ with vertices v_0 and v_1. Corollary 6.8 shows that precisely two positive edges intersect at v_1. Therefore, we can find vertex $v_2 \in \gamma_h$ different from v_0, v_1 such that $e_{v_1 v_2}$ a positive edge. This shows that $n \geq 2$. Of course $n < \infty$ because there are only finitely many vertices in $\mathcal{T}_h$.

We have identified vertices v_0, v_1 and v_2 such that $e_{v_0 v_1}, e_{v_1 v_2} \subset \gamma_h$. Suppose that we have identified vertices $v_0, v_1, \ldots, v_{k-1}$ for $k \in \{2, \ldots, n\}$ such that $e_{v_i v_{(i+1)}} \subset \gamma_h$ for each $0 \leq i \leq k-2$. We show how to identify vertex v_k such that $e_{v_{(k-1)} v_k} \subset \gamma_h$. Corollary 6.8 shows that precisely two positive edges intersect at $v_{(k-1)}$. One of them is $e_{v_{(k-2)} v_{(k-1)}}$. Let v_k be such that $e_{v_{(k-1)} v_k}$ is the other positive edge. While v_k is different from $v_{(k-2)}$ and $v_{(k-1)}$ by definition, it remains to be shown that $v_k \neq v_i$ for $0 \leq i < k-2$. To this end, note that for $1 \leq i < k-2$, we have already found two positive edges that intersect at v_i, namely $e_{v_{(i-1)} v_i}$ and $e_{v_i v_{(i+1)}}$. Therefore, it follows from corollary 6.8 that $e_{v_i v_{(k-1)}}$ cannot be a positive edge for $1 \leq i < k-2$. Hence $v_k \neq v_i$ for $1 \leq i < k-2$. On the other hand, suppose that $v_k = v_0$. Then $e_{v_0 v_{(k-1)}}$ and $e_{v_0 v_1}$ are the two positive edges intersecting at v_0. In particular, this implies that for each $0 \leq i \leq k-1$, we have found the two positive edges that intersect at vertex

v_i. Noting that $n > k-1$, let w be any vertex in γ_h different from $v_0, \dots, v_{(k-1)}$. It follows from Corollary 6.8 that $e_{v_i w}$ cannot be a positive edge for any $0 \le i \le k-1$. This contradicts the assumption that γ_h is a connected set. Hence $v_k \ne v_0$.

Repeating the above step, we identify all the distinct vertices $v_0, \dots, v_n$ in γ_h such that $e_{v_i v_{(i+1)}}$ is a positive edge for $0 \le i < n$. All vertices in γ_h can be found this way because γ_h is connected. It only remains to show that $e_{v_n v_0} \subset \gamma_h$. The argument is similar to the one given above. Corollary 6.8 shows that precisely two positive edges intersect at v_n. One of them is $e_{v_{(n-1)} v_n}$. Since $v_0, \dots, v_n$ are all the vertices in γ_h, the other edge has to be $e_{v_n v_i}$ for some $0 \le i \le n-2$. However, $e_{v_i v_n}$ cannot be a positive edge for $1 \le i < n-1$ since we have already identified $e_{v_{(i-1)} v_i}$ and $e_{v_i v_{(i+1)}}$ as the two positive edges intersecting at v_i. Hence we conclude that $e_{v_n v_0}$ is a positive edge of γ_h. $\square$

7. Injectivity on connected components of Γ_h. The main result of this section is the following lemma.

LEMMA 7.1. *Let γ and γ_h be connected components of Γ and Γ_h respectively, such that $\gamma \cap \pi(\gamma_h) \ne \emptyset$. Then $\pi : \gamma_h \to \gamma$ is a homeomorphism.*

Surjectivity of $\pi : \gamma_h \to \gamma$ in the above lemma is simple. Continuity of π over the connected set γ_h implies that $\pi(\gamma_h)$ is a connected subset of Γ. Since γ is a connected component of Γ and $\gamma \cap \pi(\gamma_h) \ne \emptyset$, $\pi(\gamma_h) \subseteq \gamma$. We also know that $\pi(\gamma_h)$ is a closed curve because γ_h is a closed curve (Lemma 6.9). Since γ is a Jordan curve, the only closed and connected curve contained in γ is either a point in γ or γ itself. In view of Lemma 5.2, $\pi(\gamma_h)$ is not a point, and hence $\pi(\gamma_h) = \gamma$

The critical step is proving injectivity. For this, we extend the result of Lemma 5.2 in Proposition 7.2 to show that π is injective over any two intersecting positive edges in γ_h (or Γ_h). This result does not suffice for an argument to prove injectivity by considering distinct points in γ_h whose images in γ coincide and then arrive a contradiction. Instead, we consider a subdivision of γ_h into finitely many connected subsets. For a specific choice of these subsets, we demonstrate using Proposition 7.2 that π is injective over each of these subsets (Proposition 7.3). Then we argue that there can be only one such subset and that it has to equal γ_h itself (Proposition 7.4).

PROPOSITION 7.2. *If e_{ap} and e_{aq} are distinct positive edges in Γ_h, then $\pi : e_{ap} \cup e_{aq} \to \Gamma$ is injective.*

Proof. Let $\alpha_i = \arccos(\hat{N}(\pi(a)) \cdot \hat{U}_{ai})$ for $i = p, q$. By Lemma 6.6, we know that $\hat{T}(\pi(a)) \cdot \hat{U}_{ap}$ and $\hat{T}(\pi(a)) \cdot \hat{U}_{aq}$ have opposite (non-zero) signs. Therefore, without loss of generality, assume that $\hat{T}(\pi(a)) \cdot \hat{U}_{ap} < 0$ and $\hat{T}(\pi(a)) \cdot \hat{U}_{aq} > 0$ so that

$$\hat{U}_{ap} = \cos\alpha_p \, \hat{N}(\pi(a)) - \sin\alpha_p \, \hat{T}(\pi(a)), \tag{7.1a}$$

$$\hat{U}_{aq} = \cos\alpha_q \, \hat{N}(\pi(a)) + \sin\alpha_q \, \hat{T}(\pi(a)). \tag{7.1b}$$

We proceed by contradiction. Suppose that x and y are distinct points in $e_{ap} \cup e_{aq}$ such that $\pi(x) = \pi(y)$. By Lemma 5.2, we know that π is injective over e_{ap} and e_{aq} respectively. Therefore, x and y cannot both belong to either e_{ap} or e_{aq}. Without loss of generality, assume that $x \in e_{ap} \setminus \{a\}$ and $y \in e_{aq} \setminus \{a\}$. In the following, we identify a point $z \in B(\Gamma, r_n)$ such that $\pi(z)$ equals both $\pi(x)$ and $\pi(a)$. This will contradict Lemma 5.2.

Let $0 < \lambda_x \le d(a,p)$ and $0 < \lambda_y \le d(a,q)$ be such that

$$x = a + \lambda_x \hat{U}_{ap}, \tag{7.2a}$$

$$\text{and} \quad y = a + \lambda_y \hat{U}_{aq}. \tag{7.2b}$$

Consider the point

$$z = \pi(x) + \xi \hat{N}(\pi(x)), \tag{7.3}$$

$$\text{where} \quad \xi = \frac{\phi(y)\lambda_x \sin\alpha_p + \phi(x)\lambda_y \sin\alpha_q}{\lambda_x \sin\alpha_p + \lambda_y \sin\alpha_q}. \tag{7.4}$$

Since λ_x, λ_y are strictly positive (by definition) and $\sin\alpha_p, \sin\alpha_q$ are strictly positive (Proposition 5.1), we know that $\lambda_x \sin\alpha_p + \lambda_y \sin\alpha_q \neq 0$. Hence z given by (7.3) is well defined. Moreover, from $|\phi(x)| \leq h_K$ and $|\phi(y)| \leq h_K$ (Proposition 4.1), it follows from (7.4) that $|\xi| \leq h_K$. Since $h_K < r_n$ by (3.6a), $z \in B(\Gamma, r_n)$. Therefore from (7.3) and Theorem 2.2, we conclude that $\pi(z) = \pi(x)$.

Next we show that $\pi(z) = \pi(a)$ as well. From Theorem 2.2 and the assumption that $\pi(y) = \pi(x)$, we have

$$x = \pi(x) + \phi(x)\hat{N}(\pi(x)). \tag{7.5a}$$

$$\text{and} \quad y = \pi(y) + \phi(y)\hat{N}(\pi(y)) = \pi(x) + \phi(y)\hat{N}(\pi(x)). \tag{7.5b}$$

Observe from (7.5) that $x \neq y \Rightarrow \phi(x) \neq \phi(y)$. Hence, subtracting (7.5b) from (7.5a) and using (7.2) yields

$$\hat{N}(\pi(x)) = \frac{x-y}{\phi(x)-\phi(y)} = \frac{\lambda_x \hat{U}_{ap} - \lambda_y \hat{U}_{aq}}{\phi(x)-\phi(y)}. \tag{7.6}$$

From (7.2a), (7.3) and (7.5a) we get

$$z = a + \lambda_x \hat{U}_{ap} + (\xi - \phi(x))\hat{N}(\pi(x)). \tag{7.7}$$

Upon using (7.1), (7.4) and (7.6) in (7.7) and simplifying, we get

$$z = a + \underbrace{\frac{\lambda_x \lambda_y \sin(\alpha_p + \alpha_q)}{\lambda_x \sin\alpha_p + \lambda_y \sin\alpha_q}}_{\zeta} \hat{N}(\pi(a)) = \pi(a) + (\phi(a) + \zeta)\,\hat{N}(\pi(a)). \tag{7.8}$$

By Theorem 2.2, (7.8) shows that $\pi(z) = \pi(a)$. Hence we have shown that $\pi(x) = \pi(a)$ (both equal point $\pi(z)$). This contradicts the fact that π is injective on e_{ap}. $\square$

To proceed, it is convenient to introduce parameterizations for γ and γ_h. To this end, consider a representation for γ_h as in (6.18), where $\{v_i\}_{i=0}^n$ are all of its vertices. From Lemma 6.9 we know that γ_h is a simple, closed curve, so let a parameterization of γ_h be $\alpha : [0,1) \to \gamma_h$ continuous and one-to-one such that

(i) $\alpha(0) = \alpha(1^-) = v_0$,

(ii) $\alpha^{-1}(v_i) < \alpha^{-1}(v_j)$ if $0 \leq i < j \leq n$,

Clearly $e_{v_i v_{(i+1)}} = \alpha[\alpha^{-1}(v_i), \alpha^{-1}(v_{i+1})]$ for $0 \leq i < n$ and $e_{v_n v_0} = \alpha[\alpha^{-1}(v_n), 1^-)$. Similarly, given that γ is a simple, closed curve, we consider a continuous and one-to-one parameterization $\beta : [0,1) \to \gamma$ of γ. As discussed at the beginning of this section, the hypotheses in Lemma 7.1 imply that $\pi(\gamma_h) = \gamma$, and in particular that $\pi(v_0) \in \gamma$. Therefore without loss of generality, we assume that $\beta(0) = \beta(1^-) = \pi(v_0)$. For future reference, we note that $\beta^{-1} : \gamma \setminus \pi(v_0) \to (0,1)$ is injective and continuous as well.

We can now define the connected subsets of γ_h alluded to at the beginning of §7. Let $P_0 := \{p \in [0,1) : \pi(\alpha(p)) = \pi(v_0)\}$. Observe that since π is injective over

each positive edge in γ_h (Lemma 5.2), each of these edges has at most one point in common with $\alpha(P_0)$. Consequently, P_0 is a collection of finitely many points. Then, noting from the definition of P_0 that $0 \in P_0$, we consider the following ordering for points in P_0:

$$P_0 = \{p_i \,:\, 0 \le i < m < \infty,\, 0 = p_0 < p_1 < \ldots < p_{m-1} < 1\}. \tag{7.9}$$

Additionally, for convenience we set $p_m = 1$. The connected subsets of γ_h we consider are the sets $\alpha([p_i, p_{i+1}))$ for $0 \le i < m$.

PROPOSITION 7.3. *For $0 \le i < m$, $\pi : \alpha[p_i, p_{i+1}) \to \gamma$ is a bijection.*

Proof. To prove the proposition, we show that the map $\psi := \beta^{-1} \circ \pi \circ \alpha$ is injective over the interval (p_i, p_{i+1}). To this end, we will need to consider the (positive) edges of γ_h contained in $\alpha[p_i, p_{i+1}]$. We denote the number of such edges by k, set $v_a = \alpha(p_i)$, and define $\{q_j\}_{j=0}^{k+1}$ as $q_j = \alpha^{-1}(v_{a+j})$. Then, by the definition of α, $\{q_j\}_{j=0}^{k+1} \subset [p_i, p_{i+1}]$ and

$$p_i = q_0 < q_1 < \ldots < q_k < q_{k+1} = p_{i+1}. \tag{7.10}$$

Notice that $k \ge 1$ because $k = 0$ would imply that π is not injective on the edge containing the points $\alpha(p_i)$ and $\alpha(p_{i+1})$, contradicting Lemma 5.2.

Consider $0 \le j \le k-1$. Proposition 7.2 shows that π is injective over $\alpha[q_j, q_{j+2}]$, and hence ψ is injective over (q_j, q_{j+2}). Since ψ is continuous over (p_i, p_{i+1}), it is continuous over (q_j, q_{j+2}) as well. Consequently, ψ is continuous and strictly monotone over (q_j, q_{j+2}).

From here, we conclude that ψ is continuous and strictly monotone over the interval $(q_0, q_{k+1}) = (p_i, p_{i+1})$. In particular, ψ is injective over (p_i, p_{i+1}). Since β^{-1} is injective over $\gamma \setminus \pi(v_0)$, we get that $\pi \circ \alpha$ is injective over (p_i, p_{i+1}), i.e., that π is injective over $\alpha(p_i, p_{i+1})$. From the definition of P_0, we know that $\pi(\alpha(p_i)) = \pi(v_0)$ and that $\pi(v_0) \notin \pi(\alpha(p_i, p_{i+1}))$. Therefore we conclude that π is in fact injective over $\alpha[p_i, p_{i+1})$.

Finally we show $\pi : \alpha[p_i, p_{i+1}) \to \gamma$ is surjective. Since π is continuous over the connected set $\alpha[p_i, p_{i+1})$, $\pi(\alpha[p_i, p_{i+1}))$ is a connected subset of γ. Since $\pi(\alpha(p_i)) = \pi(\alpha(p_{i+1})) = \pi(v_0)$, $\pi(\alpha[p_i, p_{i+1}))$ equals either $\{\pi(v_0)\}$ or γ. Injectivity of π over $\alpha[p_i, p_{i+1})$ rules out the former possibility. □

PROPOSITION 7.4. *Let P_0 be as defined in* (7.9)*. Then $P_0 = \{0\}$.*

Proof. We prove the proposition by showing that $m > 1$ yields a contradiction. Suppose that $m > 1$. For each $0 \le i < m$, let $w_i := \alpha(p_i), \gamma_h^i := \alpha[p_i, p_{i+1})$ and define $\Psi_i \colon [0,1) \to \mathbb{R}$ as $\Psi_i := \phi \circ \left(\pi\big|_{\gamma_h^i}\right)^{-1} \circ \beta$. Note that Ψ_i is well defined for each $0 \le i < m$ because $\pi : \gamma_h^i \to \gamma$ is a bijection from Proposition 7.3. Since it follows from Corollary 5.4 that $\pi^{-1} : \gamma \to \gamma_h^i$ is continuous, we get that Ψ_i is continuous for each $0 \le i < m$.

For convenience, denote $w_m = v_0 = w_0$. By definition of P_0, $\pi(w_i) = \pi(v_0)$ for each $0 \le i \le m$. From this and Theorem 2.2, we have

$$w_i = \pi(w_i) + \phi(w_i)\, \hat{N}(\pi(w_i)) = \pi(v_0) + \phi(w_i)\, \hat{N}(\pi(v_0)). \tag{7.11}$$

Since $w_i = v_0$ only for $i = 0, m$, (7.11) implies that $\phi(w_i) \ne \phi(v_0)$ for any $1 < i < m$. In particular, since $\phi(w_1) \ne \phi(v_0)$, without loss of generality, assume that $\phi(w_1) > \phi(v_0)$. Then since $\phi(w_m) = \phi(v_0)$, there exists a smallest index k such that (i) $1 \le k < m$, (ii) $\phi(w_k) \ge \phi(w_1)$ and (iii) $\phi(w_{k+1}) < \phi(w_1)$. For such a

choice of k, consider the map $(\Psi_0 - \Psi_k) : [0,1) \to \mathbb{R}$. From $\phi(w_0) = \phi(v_0)$ and $\phi(w_k) \geq \phi(w_1) > \phi(v_0)$, we get

$$(\Psi_0 - \Psi_k)(0) = \phi(w_0) - \phi(w_k) < 0. \tag{7.12}$$

On the other hand, from $\phi(w_{k+1}) < \phi(w_1)$, we get

$$(\Psi_0 - \Psi_k)(1^-) = \phi(w_1) - \phi(w_{k+1}) > 0. \tag{7.13}$$

Eqs.(7.12), (7.13) and the continuity of $\Psi_0 - \Psi_k$ on $[0,1)$ imply that there exists $\xi \in (0,1)$ such that $\Psi_0(\xi) = \Psi_k(\xi)$. For this choice of ξ, let $x_0 \in \gamma_h^0$ and $x_k \in \gamma_h^k$ be such that $\pi(x_0) = \pi(x_k) = \beta(\xi)$. That x_0 and x_k exist follows again, from Proposition 7.3. Now notice that $\Psi_0(\xi) = \Psi_k(\xi) \Rightarrow \phi(x_0) = \phi(x_k)$. Therefore from Theorem 2.2, we have

$$x_0 = \pi(x_0) + \phi(x_0)\,\hat{N}(\pi(x_0)) = \pi(x_k) + \phi(x_k)\,\hat{N}(\pi(x_k)) = x_k. \tag{7.14}$$

Eq.(7.14) shows that $\gamma_h^0 \cap \gamma_h^k \neq \emptyset$. Since γ_h is a simple curve (Lemma 6.9) and $k \neq 0$, this is a contradiction. □

Proof. [Proof of Lemma 7.1] Propositions 7.3 and 7.4 together show that $\pi : \alpha([0,1)) = \gamma_h \to \gamma$ is a bijection. Since π is continuous on γ_h, it follows from Theorem 5.3 that $\pi : \gamma_h \to \gamma$ is a homeomorphism. □

8. Connected components of Γ_h. The final step in proving part (iv) of Theorem 3.1 is the following lemma.

LEMMA 8.1. *Let γ be a connected component of Γ. Then $\gamma_h := \{x \in \Gamma_h : \pi(x) \in \gamma\}$ is a simple, closed curve.*

To prove the lemma, it suffices to show that γ_h is a connected component of Γ_h, because then Lemma 6.9 would imply that γ_h is a simple, closed curve. To this end, we consider the connected components $\{\gamma_h^i\}_{i=1}^m$ of γ_h. Clearly $m < \infty$. The objective is to demonstrate that γ_h has just one connected component, i.e., that $m = 1$. We do so in simple steps. We first show in Proposition 8.2 that each component γ_h^i is in fact a connected component of Γ_h as well. Next, we order these connected connected components according to their signed distance from γ (Proposition 8.4). Then, we inspect the relative location of triangles positively cut by each connected component with respect to the rest. This reveals that γ_h has just one connected component.

PROPOSITION 8.2. *Each connected component of γ_h is a connected component of Γ_h as well.*

Proof. Clearly Γ_h has only finitely many connected components, say $\{\Gamma_h^i\}_{i=1}^k$ for some $k < \infty$. We prove the proposition by showing that γ_h is a union of some of these components.

Suppose that $x \in \gamma_h \cap \Gamma_h^i$ for some i. From the definition of γ_h, we know $\pi(x) \in \gamma \Rightarrow \pi(\Gamma_h^i) \cap \gamma \neq \emptyset$. Since π is continuous on the connected set Γ_h^i, $\pi(\Gamma_h^i)$ is a connected subset of Γ. But $\pi(\Gamma_h^i)$ has a non-empty intersection with a connected component of Γ, namely the set γ. Consequently, $\pi(\Gamma_h^i) \subseteq \gamma$. Hence we have shown that $\Gamma_h^i \cap \gamma_h \neq \emptyset \Rightarrow \Gamma_h^i \subseteq \gamma_h$. Hence $\gamma_h = \cup_{i \in I} \Gamma_h^i$ for some $I \subseteq \{1, 2, \ldots, k\}$, which shows that γ_h is a union of connected components of Γ_h. □

In conjunction with Lemma 6.9, the above proposition shows that each connected component γ_h^i of γ_h is a simple, closed curve. Consequently, $\mathbb{R}^2 \setminus \gamma_h^i$ has precisely two connected components, namely Ω_i^- and Ω_i^+. In fact, the following proposition shows that the set Ω_i^- can always be chosen to be the component that contains

$$\omega := \{\xi - C^h h^2\,\hat{N}(\xi) \,:\, \xi \in \gamma\}, \tag{8.1}$$

where $C^h := \max_{K\in\mathcal{P}_h} C^h_K$. The purpose of such a decomposition of $\mathbb{R}^2$ is to examine the relative location of the connected components of γ_h.

PROPOSITION 8.3. *For each $1 \le i \le m$, $\mathbb{R}^2 \setminus \gamma^i_h$ has precisely two connected components Ω^-_i and Ω^+_i, such that the non-empty set ω is contained in Ω^-_i.*

Proof. Firstly, note that ω is the image of γ under a continuous map. Therefore, the assumption that γ is connected implies that ω is a connected set. Next, observe from (3.6d) that $C^h h < \tau/\sigma < 1$. Together with (3.6a), this shows that $C^h h^2 < h < r_n$. Then it follows from Theorem 2.2 that $\phi = -C^h h^2$ on ω. Since Corollary 4.5 shows that $\phi > -C^h h^2$ on Γ_h, we have $\gamma^i_h \cap \omega \subseteq \Gamma_h \cap \omega = \emptyset$. Since γ^i_h is a connected component of Γ_h (Proposition 8.2), we know from Lemma 6.9 that it is a simple, closed curve. Therefore, by the Jordan curve theorem, $\mathbb{R}^2 \setminus \gamma^i_h$ has precisely two connected components. Then $\gamma^i_h \cap \omega = \emptyset$ implies that the connected set ω is contained in one of the two connected components of $\mathbb{R}^2 \setminus \gamma^i_h$. The proposition follows from setting Ω^-_i to be the component of $\mathbb{R}^2 \setminus \gamma^i_h$ that contains ω and Ω^+_i to be the other. ∎

In the next step, we order the connected components of γ_h according to their signed distance from γ. The natural functions to consider for this ordering are the maps $\Psi_i = \phi \circ \left(\pi\big|_{\gamma^i_h}\right)^{-1}$, where $1 \le i \le m$. From Lemma 7.1 and Proposition 8.2, we note for future use that

$$\pi : \gamma^i_h \to \gamma \text{ is a homeomorphism for each } i = 1 \text{ to } m. \tag{8.2}$$

PROPOSITION 8.4. *Let $1 \le i, j \le m$. Then,*

(i) *The function Ψ_i is well defined, continuous and $-h < \Psi_i \le h$.*

(ii) *For any $\xi \in \gamma$, $\Psi_i(\xi) = \Psi_j(\xi) \iff i = j$.*

(iii) *If $\Psi_i(\xi) < \Psi_j(\xi)$ for some $\xi \in \gamma$, then $\Psi_i < \Psi_j$ on γ.*

Proof.

(i) The fact that Ψ_i is well-defined and continuous is a consequence of (8.2) and the continuity of ϕ. Part (ii) of Proposition 4.1 shows that $|\Psi_i| \le h$. From Corollary 4.5, (3.6d) and $\tau/\sigma < 1$, we also get

$$\min_{\gamma} \Psi_i = \min_{\gamma^i_h} \phi \ge -\max_{K\in\mathcal{P}_h} C^h_K h_K^2 \ge -\max_{K\in\mathcal{P}_h} C^h_K h^2 > -\frac{\tau}{\sigma} h > -h.$$

(ii) Let $\xi \in \gamma$ be arbitrary. Following (8.2), let $x_i \in \gamma^i_h$ be such that $\pi(x_i) = \xi$, where $1 \le i \le m$. From $\phi(x_i) = \Psi_i(\xi)$ and Theorem 2.2, we get

$$x_i = \pi(x_i) + \phi(x_i)\, \hat{N}(\pi(x_i)) = \xi + \Psi_i(\xi)\, \hat{N}(\xi). \tag{8.3}$$

Since $\gamma^i_h \cap \gamma^j_h = \emptyset$ for $i \ne j$, $x_i = x_j \iff i = j$. Hence (8.3) implies that $\Psi_i(\xi) = \Psi_j(\xi) \iff i = j$.

(iii) For some $i \ne j$ and $\xi \in \gamma$, assume that $\Psi_i(\xi) < \Psi_j(\xi)$. Suppose there exists $\zeta \in \gamma$ such that $\Psi_i(\zeta) \not< \Psi_j(\zeta)$. Since part (ii) shows $\Psi_i(\zeta) \ne \Psi_j(\zeta)$, we have $\Psi_i(\zeta) > \Psi_j(\zeta)$. Note that $(\Psi_i - \Psi_j)$ is a continuous map on the connected set γ. Therefore, from $(\Psi_i - \Psi_j)(\xi) < 0$, $(\Psi_i - \Psi_j)(\zeta) > 0$ and the intermediate value theorem, we know there exists $\zeta' \in \gamma$ such that $(\Psi_i - \Psi_j)(\zeta') = 0$. This contradicts part (ii). ∎

PROPOSITION 8.5. *For any $\xi \in \gamma$ and $i \in \{1, \ldots, m\}$,*

$$\ell^-_i := \left\{\xi + \lambda\, \hat{N}(\xi) \,:\, -h \le \lambda < \Psi_i(\xi)\right\} \subset \Omega^-_i. \tag{8.4}$$

Proof. Following (8.2), let $x_i \in \gamma_h^i$ be such that $\pi(x_i) = \xi$. From Theorem 2.2 and $\phi(x_i) = \Psi_i(\xi)$, we have

$$x_i = \xi + \phi(x_i)\hat{N}(\xi) = \xi + \Psi_i(\xi)\hat{N}(\xi). \tag{8.5}$$

Eq.(8.5) demonstrates that $x_i \notin \ell_i^-$ and hence that $\ell_i^- \cap \gamma_h^i = \emptyset$. Since ℓ_i^- is a connected set, this implies that either $\ell_i^- \subset \Omega_i^-$ or $\ell_i^- \subset \Omega_i^+$. Therefore, to show $\ell_i^- \subset \Omega_i^-$, it suffices to show that $\ell_i^- \cap \Omega_i^- \neq \emptyset$. To this end, consider the point $y = \xi - C^h h^2 \hat{N}(\xi)$. Observe that $y \in \ell_i^-$ because from (3.6d) and Corollary 4.5, we have

$$-h < -\frac{\tau h}{\sigma} < -C^h h^2 < \phi(x_i) = \Psi_i(\xi).$$

By definition, $y \in \omega$ as well. Since $\omega \subset \Omega_i^-$ (Proposition 8.3), we get that $y \in \ell_i^- \cap \Omega_i^-$. This shows that $\ell_i^- \cap \Omega_i^- \neq \emptyset$. □

Corollary 8.6 (of Proposition 8.5). *Let $i \in \{1, \ldots, m\}$. If v is a vertex in $\mathcal{T}_h$ such that $v \in \overline{B(\gamma, h)} \cap \Omega_i^+$, then $\phi(v) \geq 0$.*

Proof. Using $h < r_n$ (see (3.6a)) and Theorem 2.2, we know $\xi = \pi(v)$ is well defined. Following (8.2), let $x_i \in \gamma_h^i$ be such that $\pi(x_i) = \xi$. From Theorem 2.2, we have

$$y = \xi + \phi(y)\,\hat{N}(\xi) \quad \text{for } y = x_i, v. \tag{8.6}$$

Since $v \in \Omega_i^+$ and $\Omega_i^- \cap \Omega_i^+ = \emptyset$, $v \notin \Omega_i^-$. Then (8.6), $|\phi(v)| \leq h$ and Proposition 8.5 imply that

$$\phi(v) \geq \Psi_i(\xi) = \phi(x_i). \tag{8.7}$$

From (8.6) and (8.7), we have

$$d(v, x_i) = |\phi(v) - \phi(x_i)| = \phi(v) - \phi(x_i). \tag{8.8}$$

By definition of Ω_i^+, $v \in \Omega_i^+ \Rightarrow v \notin \gamma_h^i$. Therefore,

$$d(v, x_i) \geq d(v, \gamma_h^i) \geq \min_{K \in \mathcal{T}_h} \rho_K \geq \frac{1}{\sigma} \min_{K \in \mathcal{T}_h} h_K \geq \frac{\tau}{\sigma} h. \tag{8.9}$$

Then, from (8.8), (8.9) and corollary 4.5, we get

$$\phi(v) = d(v, x_i) + \phi(x_i) \geq \frac{\tau}{\sigma} h - C^h h^2 \geq 0, \tag{8.10}$$

which proves the corollary. □

Proposition 8.7. *For any $\xi \in \gamma$ and $i \in \{1, \ldots, m\}$,*

$$\ell_i^+ := \left\{\xi + \lambda\,\hat{N}(\xi) \,:\, \Psi_i(\xi) < \lambda \leq h\right\} \subset \Omega_i^+. \tag{8.11}$$

Proof. Following Lemma 7.1, let $x_i \in \gamma_h^i$ be such that $\pi(x_i) = \xi$. Note that $\ell_i^+ \cap \gamma_h^i = \emptyset$. Since ℓ_i^+ is connected, it is either contained in Ω_i^- or in Ω_i^+. Hence to prove the proposition, we assume that $\ell_i^+ \neq \emptyset$ ($\Psi_i(\xi) < h$) and demonstrate that $\ell_i^+ \cap \Omega_i^+ \neq \emptyset$.

Consider first the case in which x_i is not a vertex in γ_h^i. Let e_{ab} be the edge in γ_h^i that contains x_i. Since γ_h^i is a Jordan curve, we know that there exists $\delta > 0$ (possibly depending on x_i) such that $B(x_i, \delta) \cap \gamma_h^i$ is a connected set. Noting that $d(x_i, a), d(x_i, b) > 0$ from $x_i \in \mathbf{ri}(e_{ab})$, $h - \Psi_i(\xi) > 0$ from ℓ_i^+ being non-empty, and $h + \Psi_i(\xi) > 0$ from part (i) of Proposition 8.4, it is possible to choose $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{\delta, d(x_i, a), d(x_i, b), h \pm \Psi_i(\xi)\right\}. \tag{8.12}$$

In particular, $\varepsilon < \min\{\delta, d(x_i, a), d(x_i, b)\}$ implies that $B(x_i, \varepsilon) \cap \gamma_h^i = B(x_i, \varepsilon) \cap e_{ab}$. Hence, $B(x_i, \varepsilon) \setminus \gamma_h^i$ has precisely two connected components H_- and H_+, defined as $H_\pm = \left(B(x_i, \varepsilon) \setminus \gamma_h^i\right) \cap \Omega_i^\pm$. In particular, H_- is a convex set (being the interior of a half disc).

Let ℓ_i^- be as defined in Proposition 8.5 and set $\zeta_\pm := x_i \pm (\varepsilon/2)\,\hat{N}(\xi)$. From the definition of x_i and $\zeta_\pm$, we get

$$(\zeta_\pm - \xi) \cdot \hat{N}(\xi) = \Psi_i(\xi) \pm \frac{\varepsilon}{2}. \tag{8.13}$$

From (8.13) and $0 < \varepsilon < h - \Psi_i(\xi)$, we get $\zeta_+ \in \ell_i^+ \cap B(x_i, \varepsilon)$. Similarly, (8.13) and $0 < \varepsilon < h + \Psi_i(\xi)$ show that $\zeta_- \in \ell_i^- \cap B(x_i, \varepsilon)$. Using the latter and Proposition 8.5, we get

$$\ell_i^- \subset \Omega_i^- \Rightarrow \ell_i^- \cap B(x_i, \varepsilon) \subset H_- \Rightarrow \zeta_- \in H_-. \tag{8.14}$$

Note that $\zeta_+ \neq x_i \Rightarrow \zeta_+ \notin \gamma_h^i$. Also, $\zeta_+ \in H_-$ yields a contradiction because using $\zeta_- \in H_-$ and the convexity of H_-, we get

$$\zeta_+ \in H_- \Rightarrow \frac{1}{2}(\zeta_- + \zeta_+) \in H_- \Rightarrow \gamma_h^i \ni x_i \in H_- \Rightarrow \gamma_h^i \cap \Omega_i^- \neq \emptyset. \tag{8.15}$$

Hence we get the required conclusion that

$$\zeta_+ \in H_+ \Rightarrow \ell_i^+ \cap H_+ \neq \emptyset \Rightarrow \ell_i^+ \cap \Omega_i^+ \neq \emptyset.$$

The case in which x_i is a vertex is similar. For brevity, we only provide a sketch of the proof and omit details. By corollary 6.8, precisely two positive edges in γ_h^i intersect at x_i. Let these edges be $e_{x_i a}$ and $e_{x_i b}$. Choose ε as in (8.12) and define $H_\pm$ as done above. Define $\zeta_\pm$ as above and note that $\zeta_- \in H_-$ as done in (8.14). The main difference compared to the case when x_i is not a vertex is that now, H_- is either a convex or a concave set. If H_- is convex, arguing as in (8.15) shows that $\zeta_+ \in H_+$. To show $\zeta_+ \in H_+$ when H_- is concave, it is convenient to adopt a coordinate system. The essential step is noting that $\hat{T}(\xi) \cdot \hat{U}_{x_i a}$ and $\hat{T}(\xi) \cdot \hat{U}_{x_i b}$ have opposite (and non-zero) signs as shown by Lemma 6.6. □

Corollary 8.8. *Let $i, j \in \{1, \ldots, m\}$. If $\Psi_i(\zeta) < \Psi_j(\zeta)$ for some $\zeta \in \gamma$, then $\gamma_h^j \subset \Omega_i^+$.*

Proof. Consider any $x \in \gamma_h^j$ and let $\xi = \pi(x)$. Proposition 8.4 shows that

$$\Psi_i(\xi) < \Psi_j(\xi) = \phi(x) \leq h \tag{8.16}$$

Using $x = \xi + \phi(x)\hat{N}(\xi)$ (from Theorem 2.2) and (8.16) in Proposition 8.7 shows that $x \in \Omega_i^+$. Since $x \in \gamma_h^j$ was arbitrary, the corollary follows. □

Proposition 8.9. *Let $i, j \in \{1, \ldots, m\}$ and $K = (a, b, c) \in \mathcal{P}_h$ have positive edge $e_{ab} \subset \gamma_h^j$. If $\gamma_h^j \subset \Omega_i^+$, then $\overline{K} \subset \Omega_i^+$.*

Proof. Note that $\gamma_h^j \subset \Omega_i^+$ immediately implies $i \neq j$. First, we show that $K_h^i := \overline{K} \cap \gamma_h^i = \emptyset$. Since γ_h^i is a collection of positive edges, K_h^i is either empty, or a vertex of K or an edge of K. From $i \neq j$, we get

$$e_{ab} \cap \gamma_h^i \subseteq \gamma_h^j \cap \gamma_h^i = \emptyset. \tag{8.17}$$

Therefore, neither a nor b belong to K_h^i. Hence K_h^i does not contain any edge of K. Since every vertex in γ_h^i has $\phi \geq 0$ but $\phi(c) < 0$, $c \notin K_h^i$. Hence we conclude that $K_h^i = \emptyset$.

Since $\overline{K}$ is a connected set and $\overline{K} \cap \gamma_h^i = \emptyset$, either $\overline{K} \subset \Omega_i^+$ or $\overline{K} \subset \Omega_i^-$. However, $e_{ab} \subset \gamma_h^j \subset \Omega_i^+$ shows that $\overline{K} \cap \Omega_i^+ \neq \emptyset$. Hence $\overline{K} \subset \Omega_i^+$. □

Proof. [Proof of Lemma 8.1] We need to show that γ_h has just one connected component, i.e., that $m = 1$.

Suppose that $m > 1$. Then γ_h^1 and γ_h^2 are connected components of γ_h. Consider any $\xi \in \gamma$. Proposition 8.4 shows that $\Psi_1(\xi) \neq \Psi_2(\xi)$. Therefore, without loss of generality, assume that $\Psi_1(\xi) < \Psi_2(\xi)$. Then Corollary 8.8 shows that $\gamma_h^2 \subset \Omega_1^+$.

Consider any positive edge $e_{ab} \subset \gamma_h^2$. By definition, there exists a vertex c in $\mathcal{T}_h$ with $\phi(c) < 0$ such that $K = (a, b, c) \in \mathcal{P}_h$. Then using $\gamma_h^2 \subset \Omega_1^+$ in Proposition 8.9 shows that $\overline{K} \subset \Omega_1^+$. In particular, this implies that $c \in \Omega_1^+$. However, c being a vertex in Ω_1^+ with $\phi(c) < 0$ contradicts Corollary 8.6.

Therefore $m = 1$ and hence $\gamma_h = \gamma_h^1$. Hence Proposition 8.2 shows that γ_h is a connected component of Γ_h. In turn, Lemma 6.9 implies that γ_h is a simple, closed curve. □

Proof of Theorem 3.1. The theorem follows essentially from compiling results we have proved thus far.

(i) See Lemma 6.3.

(ii) For a positive edge $e \subset \Gamma_h$, Lemma 5.5 shows that π is C^1 on $\mathbf{ri}\,(e)$ with the Jacobian bounded away from zero. The inverse function theorem then implies that π is in fact a C^1-diffeomorphism on $\mathbf{ri}\,(e)$.

(iii) See Corollary 4.5 for lower bound of ϕ and Proposition 4.1 for the upper bound. See Lemma 5.5 for the bounds for the Jacobian. In fact, it holds that

$$0 < \frac{\sin\theta_c}{(\pi - \theta_c)(1 + Mh_0)} \left(\cos\Theta_c - \frac{Mh_0\sigma}{1 - Mh_0} \right) < \frac{\sin(\beta_K - \vartheta_K)}{1 + M_K h_K} < J \tag{8.18}$$

for any $h < h_0$, where h_0 satisfies (3.8). This shows that J is bounded away from zero independently of h. To get it we note that

$$\cos\beta_K \leq C_K^h \sigma_K h_K < \frac{Mh_0\sigma}{1 - Mh_0} < \cos\Theta_c \leq \cos\vartheta_K. \tag{8.19}$$

We can then conclude that

$$0 < \cos\Theta_c - \frac{Mh_0\sigma}{1 - Mh_0} < \cos\vartheta_K - \cos\beta_K < \beta_K - \vartheta_K \leq \sin(\beta_K - \vartheta_K)\frac{\pi - \theta_c}{\sin\theta_c}, \tag{8.20}$$

where we have used that $\beta_K - \vartheta_K < \pi - \theta_c$ and that $\sin(x)/x \geq \sin\theta_c/(\pi - \theta_c)$ for $0 \leq x < \pi - \theta_c$, and (8.18) follows.

(iv) Consider any connected component γ of Γ and define $\gamma_h = \{x \in \Gamma_h : \pi(x) \in \gamma\}$. Lemma 8.1 shows that γ_h is a connected set and that it is a simple, closed curve. Then from Lemma 7.1, we get that $\pi : \gamma_h \to \gamma$ is a homeomorphism. It then follows that $\pi : \Gamma_h \to \Gamma$ is a homeomorphism.

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